Computers and Structures 104–105 (2012) 38–43

Contents lists available at SciVerse ScienceDirect

Computers and Structures

journal homepage: www.elsevier.com/locate/compstruc

The boundedness of Gorman's Superposition method for free vibration analysis of plates

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ARTICLE INFO

Article history:
Received 7 January 2011
Accepted 13 February 2012
Available online 31 March 2012

Keywords:
Superposition method
Vibration
Natural frequencies
Upper bound
Lower bound
Plate

ABSTRACT

Gorman's Superposition method is known as one of the most efficient methods to solve the eigenvalue problems of plates because of its excellent convergence rate. However, there are few published results available that provide sufficient information on its boundedness. Here we have considered the nature of convergence of the eigenvalues for rectangular plates with the following sets of boundary conditions, completely free, fully clamped and cantilever. This paper shows numerically, the boundedness of the Superposition method for undamped vibration problems of rectangular isotropic plates subjected to different boundary conditions. The Superposition method gives upper bound results for eigenvalues of plates if the building blocks used in the Superposition method are subjected to stiffer boundary conditions than those of the original system being modelled. In contrast, the Superposition method yields lower bound results if the boundary conditions of building blocks are more flexible than those of the original system. The results would be useful to estimate the maximum possible error if the other bound can be obtained by another method.

1. Introduction

Much research has been conducted into eigenvalue problems such as plate vibration problems, using a wide range of methods. For only specific conditions, exact closed form solutions are available, and otherwise the problems are solved approximately. It is known that the Rayleigh–Ritz method always gives upper bounds for the eigenvalues, which means that the actual eigenvalues would be less than or equal to the obtained values. However, the error due to discretisation cannot be calculated easily. Thus it is useful to know when the Superposition method gives lower bound results to the same problem, so that the cases for which the actual eigenvalues could be delimited between these upper and lower bounds will be known. We have considered combinations of different boundary conditions to study this.

The Superposition method developed by Gorman has been successfully applied for the analysis of undamped out-of-plane vibrations of single isotropic plates [1]. Gorman's Superposition method solves a given plate problem by superimposing the steady state response of plate subsystems subject to different boundary conditions and driven along one edge by a distributed force, moment, translation or rotation, which are referred to as building blocks [1,2]. The method has also been applied to analyse more complicated systems, such as orthotropic plates, plates with elastic supports, point supported plates, triangular and parallelogram plates, plates under in-plane forces, Mindlin plates and laminated plates, as well as in-plane vibrations of plates [2–4]. The Superposition method was applied to not only plates but also open cylindrical shells [5]. Recently, this method was also shown to be applicable for the determination of steady state response of plates [6]. The Superposition method may be one of the most efficient methods to solve the eigenvalue problems of plates because of its excellent convergence rate [7–9]. However, there are few published results available that provide sufficient information for the boundedness of the Superposition method.

The interesting question whether the Superposition method gives bounded results is raised by Ilanko [10]. He predicts that whether it gives upper bound or lower bound results for the natural frequencies depends on the boundary conditions of the actual plate and those of the building blocks used. In cases where the building blocks used in the Superposition method is subject to stiffer boundary conditions than those of the original system being modelled, it gives upper bound results. This would be the case where completely free plates are modelled by using building blocks with slip-shear boundary conditions. On the contrary, it gives lower bound results when the building blocks are subject to more flexible conditions at the boundaries. This would be the case where fully clamped plates are modelled by using the building blocks with simply supported boundary conditions.

The author has done a comprehensive free vibration analysis using the Superposition method on the completely free plates

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http://dx.doi.org/10.1016/j.compstruc.2012.03.009

and the fully clamped plates for various aspect ratios, and convergence tests were carried out thoroughly for the first 12 modes [7]. The research is now extended to plates with combinations of free and clamped edge conditions, for example cantilever plates. The investigation shows, for the first time, that the boundedness of the Superposition method, which is predicted in reference [10], is numerically confirmed for undamped vibration of single isotropic plates under various boundary conditions.

2. The Superposition method

2.1. The completely free plate

Free vibration analysis of plates using the Superposition method is described in detail, in Refs. [1,2]. The essential steps in the derivations are presented here for completeness.

Consider the motion of the rectangular plate with the dimensions a and b as shown in Fig. 1. The partial differential equation governing the out-of-plane vibration of rectangular plates is expressed in non-dimensional form using dimensionless coordinates ξ and η, where $\xi = x/a$, $\eta = y/b$. The equation is written as

$$\frac{\partial^4 W(\xi,\eta)}{\partial\eta^4} + 2\Phi^2\frac{\partial^4 W(\xi,\eta)}{\partial\eta^2\partial\xi^2} + \Phi^4\left\{\frac{\partial^4 W(\xi,\eta)}{\partial\xi^4} - \lambda^4 W(\xi,\eta)\right\} = 0 \quad (1)$$

where $\lambda^2 = \omega a^2\sqrt{\rho h/D}$, W: dimensionless plate lateral displacement, Φ: the plate aspect ratio b/a, D: plate flexural rigidity $(Eh^3/12)/(1-\nu^2)$, E: elastic modulus of the material, h: thickness of plate, ρ: density of plate, and ν: Poisson's ratio.

In the Superposition method, the plate is considered [2] as consisting of four building blocks that have exact solutions. Fig. 2 shows the building blocks used for the analysis of a completely free plate. The two small adjacent circles depict slip-shear condition, which is that there is no rotation normal to the edge and no vertical edge reaction. The rotation, R, is applied on a driving edge of each building block. The displacement of the original plate, $W(x,y)$, is expressed as the sum of the displacement of the sub-systems (Eq. (2)).

$$W(\xi,\eta) = W_1 + W_2 + W_3 + W_4 \quad (2)$$

The displacements of the first building block is taken in the form of a Lévy type solution,

$$W_1(\xi,\eta) = \sum_{m=0,1\ldots}^{k} Y_m(\eta)\cos m\pi\xi \quad (3)$$

The edge rotation along the edge $\eta = 1$ is expressed as following Fourier expansion

$$\frac{\partial W_1(\xi,\eta)}{\partial\eta} = \sum_{m=0,1\ldots}^{k} E_m\cos m\pi\xi \quad (4)$$

By enforcing the boundary condition of zero vertical edge reaction and the equilibrium of edge rotation, the analytical function $Y_m(\eta)$ is readily determined. The solutions for $Y_m(\eta)$ are expressed in terms of the coefficients E_m, in Ref. [2] as for $\lambda^2 > (m\pi)^2$

$$Y_m(\eta) = E_m(\theta_{m11}\cosh\beta_m\eta + \theta_{m12}\cos\gamma_m\eta) \quad (5)$$

where

$$\theta_{m11} = 1/\{(\beta_m - ZZ_1\gamma_m/ZZ_2)\sinh\beta_m\}$$

and

$$\theta_{m12} = ZZ_1/\{ZZ_2(\beta_m - ZZ_1\gamma_m/ZZ_2)\sin\gamma_m\}$$

in which

$$ZZ_1 = -\beta_m\{\beta_m^2 - (2-\nu)\Phi^2(m\pi)^2\}$$

and

$$ZZ_2 = \gamma_m\{\gamma_m^2 + (2-\nu)\Phi^2(m\pi)^2\}$$

and, for $\lambda^2 < (m\pi)^2$

$$Y_m(\eta) = E_m(\theta_{m21}\cosh\beta_m\eta + \theta_{m22}\cosh\gamma_m\eta) \quad (6)$$

where

$$\theta_{m21} = 1/\{(\beta_m - ZZ_3\gamma_m/ZZ_4)\sinh\beta_m\}$$

and

$$\theta_{m22} = ZZ_3/ZZ_4\{(\beta_m + ZZ_3\gamma_m/ZZ_4)\sinh\gamma_m\}$$

in which

$$ZZ_3 = -\beta_m\{\beta_m^2 - (2-\nu)\Phi^2(m\pi)^2\}$$

and

$$ZZ_4 = \gamma_m\{\gamma_m^2 - (2-\nu)\Phi^2(m\pi)^2\}$$

where

$$\beta_m = \Phi\sqrt{\lambda^2 + (m\pi)^2}$$

and

$$\gamma_m = \Phi\sqrt{\lambda^2 - (m\pi)^2} \quad \text{or} \quad \gamma_m = \Phi\sqrt{(m\pi)^2 - \lambda^2},$$

whichever is real.

The solution to the second building block can be easily generated from the first building block by interchanging the variables η and ξ. The aspect ratio must replaced by the inverse of the aspect ratio and λ^2 must be multiplied by Φ^2. The subscript needs to be changed from m to n. Once the solutions to the first and second building blocks are obtained, solutions to the third and forth building blocks are determined by simply replacing η in the first building block solution to $1-\eta$, ξ in the second building block solution to $1-\xi$, and changing subscripts to p and q respectively.

2.2. The fully clamped plate

The modes for fully clamped plates are obtained in a similar manner to the procedure adopted for the completely free plates. The building blocks in Fig. 2 are replaced by the plates whose all edges are simply supported. Their driving edges are subjected to bending moments instead of the edge rotation. The solutions of the first building block utilised for the fully clamped plate are given by the following equations [1]:

$$W_1(\xi,\eta) = \sum_{m=1,2\ldots}^{k} Y_m(\eta)\sin m\pi\xi \quad (7)$$

for $\lambda^2 > (m\pi)^2$

$$Y_m(\eta) = E_m(\theta_{m11}\sinh\beta_m\eta + \theta_{m12}\sin\gamma_m\eta) \quad (8)$$

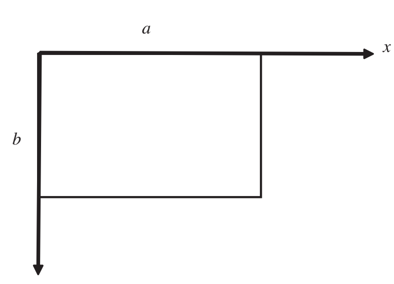

Fig. 1. A rectangular plate.