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Variational asymptotic modeling of composite beams with spanwise heterogeneity ☆

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ABSTRACT

This paper is concerned with modeling composite beams with spanwise heterogeneity. We first formulate the original three-dimensional problem in an intrinsic form which admits a geometrically exact formulation. Taking advantage of slenderness of beam structure and smallness of heterogeneity, we use the variational asymptotic method to systematically obtain an effective beam model through simultaneous homogenization and dimensional reduction. This approach is implemented in the commercial code VABS using the finite element technique for the purpose of dealing with composite beams with spanwise heterogeneity in real applications. A few examples are used to demonstrate the capability of this new model.

1. Introduction

Along with the rapidly increasing popularity of composite materials and structures, research on accurate and general modeling of structures made of them has remained as a very active field in the last several decades. Moreover the increased knowledge and fabrication techniques of them are possible to manufacture new materials and structures with optimized microstructures to achieve the ever-increasing performance requirements. Although it is logically sound to use the well-established finite element analysis (FEA) to analyze such materials and structures by meshing all the details of constituent microstructures, it is not a practical and efficient way, which requires an inordinate number of degrees of freedom (i.e. computing cost) to capture the micro-scale behavior.

If the heterogeneous structure is considered as a periodic assembly of many unit cells (UCs), which is the building block of the heterogeneous material, and the size of UC (d) is much smaller than the size of the structure (L) (i.e. $\eta = d/L \ll 1$), it is possible to homogenize the heterogeneous UC with a set of effective material properties through a micromechanical analysis of the UC. With these effective properties, the analyst can replace the original heterogeneous structure with a homogeneous one and carry out structural analysis for global behavior. In the past several decades, numerous micromechanical approaches have been suggested in the literature, such as the self-consistent model [1–3], the variational approach [4,5], the method of cells [6–9], recursive cell method [10], mathematical homogenization theories [11–13], finite element approaches using conventional stress analysis of a representative volume element [14], variational asymptotic method for unit cell homogenization (VAMUCH) [15,16], and many others (see, e.g. [17–21] for a review).

Many composite structures in real applications are also dimensionally reducible structures [22] with one or two dimensions much smaller than others. Composite beam structure is an example with the cross-sectional dimension h much smaller than the axial dimension (i.e. $e = h/L \ll 1$). If there are still many unit cells along the cross sectional directions (i.e. $\eta \ll e$), we can use the traditional two-step approach that performs homogenization using micromechanics first to obtain effective properties of the heterogeneous material, then performs dimensional reduction to construct a beam model for structural analysis. Usually, composite beams do not have many unit cells in the cross-sectional plane. For example, most sandwich beams only have many repetitive unit cells along the longitudinal direction. That is, the periodicity is exhibited only longitudinally and we have either $e \ll \eta$ or $e \sim \eta$. As pointed out by Kohn and Vogelius [23] for periodic plates if $e \ll \eta$, the order of the aforementioned two-step approach should be reversed. That is, we need to carry out the dimensional reduction to construct plate models first, then homogenize the heterogeneous surface with periodically varying plate properties. If $e \sim \eta$, the two steps in the two-step approach should be performed at the same time, that is, both small parameters (e and η) should be considered during modeling of such structures. And several studies have shown that models considering e and η simultaneously also give accurate results for the case $e \ll \eta$ [24–26].

In recent years, the formal asymptotic method has been used to study the case of periodic beams [27,28,25]. It is a modification to the asymptotic homogenization method which is a direct

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application of the formalism of two scales to the original three-dimensional (3D) equations governing the beam structure. However, although these models are mathematically elegant and rigorous without introducing *ad hoc* assumptions, it is not easy to relate the equations derived using this method with simple engineering models and extend this approach to geometrical nonlinear problems. Last but not least, it is difficult to implement these theories numerically.

As a remedy to the shortcomings of formal asymptotic method, we propose to use the variational asymptotic method (VAM) [29] to construct effective beam models for these structures through simultaneous homogenization and dimensional reduction. First, the 3D anisotropic elasticity problem is formulated in an intrinsic form suitable for geometrically nonlinear analysis. Then, considering both e and η, we use VAM to rigorously decouple the original 3D anisotropic, heterogeneous problem into a nonlinear one-dimensional (1D) beam analysis on the macroscopic level and a linear 3D unit cell analysis with only axial periodicity on the microscopic level. The unit cell problem can be easily implemented using the finite element technique for numerically obtaining the effective beam constants for the 1D beam analysis and recovering the local displacement, strain, and stress fields based on the macroscopic behavior. Several examples will be used to demonstrate the application and accuracy of this new model and the companion code VABS.

2. Three-dimensional formulation

In the 3D space, a beam can be modeled by a reference line r measured by the axial coordinate x_1, and by a typical cross section A with h as its characteristic dimension. A can be described by cross-sectional Cartesian coordinates x_α (here and throughout the paper, Greek indices assume values 2 and 3 while Latin indices assume 1, 2, and 3. Repeated indices are summed over their range except where explicitly indicated). Let us now consider a heterogeneous beam formed by periodic repetition of a certain UC (Ω) over the beam axial coordinate x_1 along r (see Fig. 1). To describe the rapid change in the material characteristics in the axial direction, we need to introduce one so-called 'fast' coordinate y_1 parallel to x_1. These two sets of coordinates are related as $y_1 = x_1/\eta$.

If the UC is a cuboid as depicted in Fig. 1, we can describe the domain (Ω) occupied by the UC using y_1 and x_α as

$$\Omega = \left\{ (y_1, x_2, x_3) \middle| -\frac{d_1}{2} < y_1 < \frac{d_1}{2}, -\frac{a_2}{2} < x_2 < \frac{a_2}{2}, -\frac{a_3}{2} < x_3 < \frac{a_3}{2} \right\} \tag{1}$$

As our goal is to homogenize the heterogenous beam, we need to assume that the exact solutions of the field variables have volume averages over Ω. For example, if $u_i(x_1,x_2,x_3;y_1)$ are the exact displacements within the UC, there exist $v_i(x_1)$ such that

$$v_i = \frac{1}{\Omega}\int_{y_1}\int_{x_2}\int_{x_3} u_i dy_1 dx_2 dx_3 = \frac{1}{\Omega}\int_{\Omega} u_i d\Omega \equiv \langle u_i \rangle \tag{2}$$

Due to the existence of a distinct scale separation between two types of spatial variations described by y_1 and x_1, the derivative of a function u_i defined in Ω with respect to x_1 can be evaluated as

$$\frac{\partial u_i(x_1,x_2,x_3;y_1)}{\partial x_1} = \frac{\partial u_i}{\partial x_1}\Big|_{y_1=const} + \frac{1}{\eta}\frac{\partial u_i}{\partial y_1}\Big|_{x_1=const} \equiv u_i' + \frac{1}{\eta}u_{i|1} \tag{3}$$

Note that in real calculation, η is not a number but denoting the order of the term it is associated with.

Letting $\mathbf{b}_i$ denote an orthonormal triad tangent to x_i for the undeformed structure, one can then describe the position of any material point along r by its position vector $\hat{\mathbf{r}}$ relative to a point O fixed in an inertial frame, such that

$$\hat{\mathbf{r}}(x_1,x_2,x_3) = \mathbf{r}(x_1) + x_\alpha \mathbf{b}_\alpha(x_1) \tag{4}$$

where $\mathbf{r}$ is the position vector from O to the point of the reference line and $\mathbf{r}' = \mathbf{b}_1$.

When the beam deforms, the particle that had position vector $\hat{\mathbf{r}}$ in the undeformed state now has position vector $\widehat{\mathbf{R}}$ in the deformed configuration, such as

$$\widehat{\mathbf{R}}(x_i;y_1) = \mathbf{R}(x_1) + x_\alpha \mathbf{T}_\alpha(x_1) + w_i(x_1,x_2,x_3;y_1)\mathbf{T}_i(x_1) \tag{5}$$

where $\mathbf{R}$ denotes the position vector of the reference line for the deformed structure, $\mathbf{T}_i$ forms a new orthonormal triad for the deformed beam configuration and $\mathbf{T}_1$is specifically tangent to the deformed reference line, and w_i are the warping functions, which are introduced to accommodate all possible deformation other than

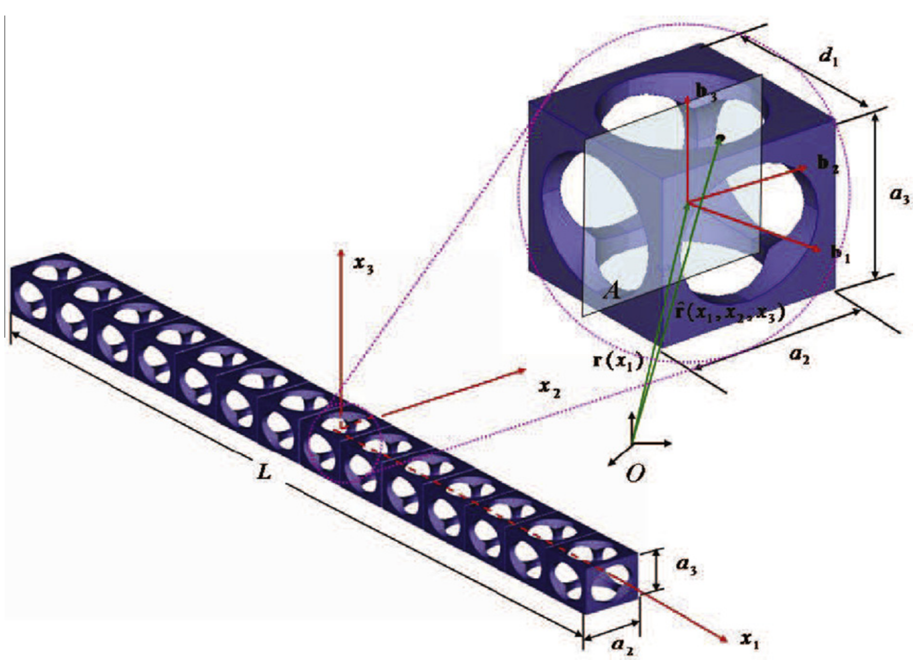

Fig. 1. A heterogeneous beam with representative periodicity cell.