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Virtual functions of the space–time finite element method in moving mass problems

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ABSTRACT

Classical time integration schemes fail in vibration analysis of complex problems with moving concentrated parameters. Moving mass problems and moving support problems belong to this group. Commercial systems of dynamic simulations do not support such an analysis. Moreover, the classical finite element method with the Newmark-type time integration method does not allow us to obtain convergent results at all. The reason lies in the impossibility of full mathematical consideration of the time integration stage and the analysis of inertial terms of a travelling mass. Both of them, unfortunately, are decoupled. In this paper we propose an efficient and exact numerical approach to the problem by using the space–time finite element method. We derive characteristic matrices of the discrete element of the string and the Bernoulli–Euler beam that carry the concentrated mass. We present four types of virtual functions in time and we apply two of them to the practical analysis. Displacements in time obtained numerically are compared with semi-analytical results. Almost perfect coincidence proves the efficiency of the approach.

1. Introduction

The paper deals with the numerical approach to the problem of structural vibrations under a travelling inertial load. Travelling non-inertial loads are unlikely to be solved by commercial codes. Such problems are not implemented in most of them. Inertial loads are not implemented in computer systems at all. Problems with travelling masses are of special interest in engineering practice. The influence of the mass attached locally to the structure cannot be neglected. We can only mention here the coupling of the 500 kg mass of the train wheel with a rail or a track. A similar case occurs in problems concerning railway power collectors. The speed of the rail vehicle can reach the critical value. In such a case the wave response significantly differs from the response of massless systems.

In the paper we present an algorithm for the moving mass analysis in the case of unidimensional structures: a string or a bar and the Bernoulli–Euler beam. In the case of other types of structures the approach is identical. We derive and list the matrices explicitly. The resulting characteristic matrices can be directly applied to numerical algorithms. The principle of application of the space–time finite element method to the problem with inertial travelling load was presented by Bajer and Dyniewicz [1] who showed the way from the differential equation to the numerical scheme and the step-by-step formula by use of the space–time element method. The solution was limited to the simplest problem of string vibrations and to the use of the Dirac delta function as a virtual distribution of the velocity. The quality of the solution could, however, be improved by the application of modified virtual functions in the formulation. This paper will describe the solution of the problem with higher accuracy formulas and apply them to more complex structures–beams.

The classical finite element approach to the moving mass problems with the Newmark time integration method fails. The difficulty lies in the methodology of the solution of the variable coefficients differential equation with the classical time integration method. In this case the spatial discretisation is performed at a selected time point t_i. Vertical acceleration is expressed in the travelling point $x = vt$. The solution is obtained by introduction of the so-called Renaudot formula, which in fact is the chain-rule derivative of the vertical displacement. Thus the acceleration in the inertial term, for $x = vt$, results in three terms

$$\frac{\mathrm{d}^2u(vt,t)}{\mathrm{d}t^2} = \left.\frac{\partial^2u(x,t)}{\partial t^2}\right|_{x=vt} + 2v\left.\frac{\partial^2u(x,t)}{\partial x\partial t}\right|_{x=vt} + v^2\left.\frac{\partial^2u(x,t)}{\partial x^2}\right|_{x=vt}, \qquad (1)$$

interpreted as the vertical acceleration, the Coriolis acceleration and the centrifugal acceleration, respectively. The direct use of (1) in the differential equation governing the motion of the continuous structure results in wrong formulas, since this mathematical step is executed rather automatically, based on two separate mathematical stages: construction of the time integration scheme and contribution of the moving mass term based on (1). Then characteristic matrices, i.e. mass, damping, stiffness, etc., are established. They are related to time t_i and do not contribute properly to the influence of terms with variable coefficients.

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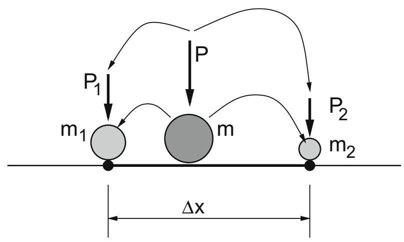

Fig. 1. Ad hoc moving mass lumping in nodes.

A simple ad hoc mass splitting between neighbouring nodes (Fig. 1) results in divergence as well. In some cases, especially in beams, numerical solutions are limited, but very inaccurate. In the case of string vibrations, governed by a purely hyperbolic differential equation, such a strategy results in divergent solutions. The ad hoc lumping can be applied in one particular case only: the mass must be replaced from node to node as a whole value. In practice, the mesh must be sized in relation to the time step and mass velocity: $\Delta x = v\Delta t$. It makes the solutions useless. We are sure that classical time integration methods would result in correct formulas if they were preceded by a correctly performed analysis. Unfortunately, up till now the authors have not been successful.

Contrary to the classical approach, the space–time finite element method allows us to perform consequently the solution of the variable coefficient differential equation in time interval $[t_i, t_{i+1}]$. The time stepping formula is derived together with the analysis of the travelling mass vertical acceleration. This last feature requires a more complex mathematical analysis. The typical approach to the space–time element method with a Dirac delta virtual function allows us to derive characteristic matrices in the step-by-step procedure. In this case, however, the product of Dirac delta virtual function and the second Dirac delta function describing the concentrated moving mass must be integrated over space and time. Although the resulting time stepping scheme is unconditionally stable with respect to the time step, the accuracy for longer time steps can be insufficient. Below, we consider other virtual functions which result in relatively simple interpretation and ensure higher-order accuracy.

The space–time finite element approach differs from the classical finite element method. First of all, in a classical approach the spatial and temporal discretisation are carried on separately. The space domain of the structure is discretized, for example, by the finite element method, finite difference method, boundary element method, etc. Time integration is performed by a difference method. The Newmark method or a derivative method, i.e. the central difference scheme and trapezoidal rule, is usually applied at this stage. Well-known classical methods of integration of the differential equations like Runge–Kutta methods, Adams methods and others can also be placed in this group. A classical approach to the vibration analysis of the structure can shortly be written by relations which describe the global (i.e. both in space and in time) interpolation of fundamental quantities

$$q(\mathbf{x}, t) = \mathbf{N}(\mathbf{x})\mathbf{T}(t)\mathbf{q}_e. \tag{2}$$

$\mathbf{N}(\mathbf{x})$ is the interpolation formula applied to space, for example, shape functions in the FEM, and $\mathbf{T}(t)$ is a time interpolation of the nodal quantity $\mathbf{q}_e = [\mathbf{q}_i, \mathbf{q}_{i+1}]^T$ in two while limiting the time interval $[t_i, t_{i+1}]$. Let us examine the uncoupling of both functions. The space–time finite element approach is described by the following interpolation:

$$q(\mathbf{x}, t) = \mathbf{N}(\mathbf{x}, t)\mathbf{q}_e. \tag{3}$$

$\mathbf{N}(\mathbf{x}, t)$ is the matrix interpolation function defined in a space–time subdomain (Fig. 2). We emphasise here that the form of Eq. (3) is

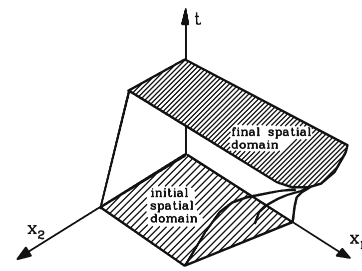

Fig. 2. Space–time subdomain.

more general than (2) and the classical finite element approach can be considered as a particular case of the space–time element method. In the space–time approach a non-stationary discretisation can also be used. In the case of a stationary mesh and in problems without damping, one can write a pass from one approach to another. Characteristic matrices, however, differ. In a general case both approaches differ. This also occurs in the case of spatial elements carrying the travelling mass. Here the second fundamental difference must be emphasised: the finite element approach uses the difference schemes for time integration while the space–time approach uses the integral formulas in formulation of the resulting time stepping schemes.

We have said that the string solution diverges even at low velocity range and with small ratio of the moving mass to the span mass. In Fig. 3 the moving mass to string mass ratio was equal to 0.1 and the mass velocity was below 0.2 of the wave speed in the string. In practice these values are relatively low. Real problems require both parameters to be even greater than one. We should be able to simulate the following technical problems: vibrations of railway tracks, vehicle passing over bridges, pantograph collectors in railways, magnetic railways, guideways in robotic technology, gun barrel, airfield plates, etc. In the case of a beam or a plate, numerical solutions are usually limited, because of parabolic terms in the differential equation. They are, however, highly inaccurate. Several papers deal with the discrete analysis of the moving mass problem [2–4]. Unfortunately, the authors do not present numerical results obtained for the inertial load. A simple massless force or oscillator is used in their demonstration, or theoretical and experimental results are only compared. All of the so-called mass forces finally are replaced by massless loads. What is more, the analytical derivations do not consider correctly the fundamental inertial term

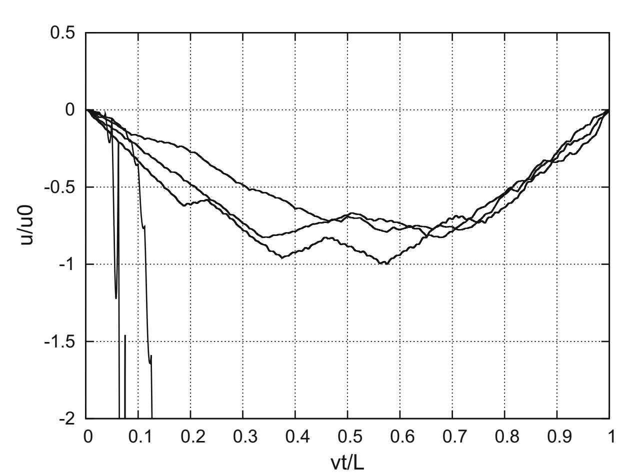

Fig. 3. Divergence of the existing numerical solutions.