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Computers and Structures 83 (2005) 2368–2380
Computers & Structures
www.elsevier.com/locate/compstruc

Analytical integration of stress field and tangent material moduli over concrete cross-sections

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Received 25 November 2003; accepted 17 March 2005
Available online 24 August 2005

Abstract

This paper presents a novel stress field and tangent material moduli integration procedure over a cross-section of a biaxially loaded concrete beam. The procedure assumes a sufficiently simple analytical form of the constitutive law of concrete, the polygonal shape of the boundary of the simply- or multi-connected cross-section and the monotonically increasing loading. The area integrals are transformed into the boundary integrals and then integrated analytically. The computational efficiency of the procedure is analyzed by comparing it with respect to the number of floating-point operations needed in various numerical integration-based methods. It is found that the procedure is not only exact, but also computationally effective.

Keywords: Reinforced concrete section; Biaxial bending; Analytical integration

1. Introduction

The non-linear finite-element analysis of reinforced concrete spatial beams and frame structures requires the integration of stresses and tangent material moduli over the cross-section. Since the governing equations of these structures are non-linear and must therefore be solved iteratively, the integrals over the cross-sections need to be evaluated many times. Thus, it is of great importance for us to be able to evaluate cross-sectional integrals as efficiently as possible. Since the area of the reinforcing steel bars is relatively small compared to the area of concrete, we may assume a constant stress field across each steel bar, which makes the integration over the steel bars very simple. The difficult part of the reinforced concrete section analysis is thus the integration of the stress field and the tangent material moduli of concrete.

A number of numerical methods have been proposed in order to make the integrations more efficient, see, e.g. [2,8,10]. The methods presented by Bonet et al. and Fafitis are particularly convenient when the stress field varies only in one direction. Their methods use Green's Theorem and transform the area integral into the boundary integral, which is then integrated numerically. While such an approach is more efficient than the one using directly the area integrals, it is still not computationally optimal due to the fact that the numerical integration inherently introduces errors. Moreover, the error in the cross-sectional integrals might imply a substantial error in force–deflection curves near the ultimate

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0045-7949/$ - see front matter © 2005 Elsevier Ltd. All rights reserved.
doi:10.1016/j.compstruc.2005.03.030

load [12]. The error can, clearly, be made smaller by increasing the number of integration points, yet this increases the computational time and, consequently, reduces the time–efficiency of the overall finite-element algorithm. To make the integration procedure both time-efficient and exact, we have developed an analytical integration scheme as described in the sequel.

For the deduction of an analytical integration algorithm, the constitutive law of concrete needs to be prescribed in an analytical form. We have chosen the constitutive law proposed by Desayi and Krishnan [4] for the concrete in compression, and that of Bergan and Holand [1] for the concrete in tension. The next assumption concerns the strain distribution over the cross-section. We follow the standard approach in reinforced concrete beam analysis and assume the linear strain distribution (see, e.g. [3,7,9,11]). For the linear strain distribution, it is easy to find a constant strain and stress direction. With the help of some change of integration variables and by the use of Green's Theorem, we transform area integrals into the path integrals along the boundary of the cross-section of known, analytically integrable functions. If the cross-section, possibly hollow, can be approximated by a polygon (which is often the case in practice), an efficient formula for the analytical integration follows. We assume a monotonic increase in strains with the increase of a load, and thus disregard strain-reversals at any point of the cross-section. This assumption limits the applicability of the present procedure to the analyses of the ultimate limit capacity and the serviceability state of a frame structure.

The exactness of an analytical approach is obvious, while its computational efficiency might be doubtful if the final analytical expressions become very cumbersome. We demonstrate the efficiency of the present method via three numerical examples, in which we compare the accuracy and the required number of floating-point operations with several numerical integration-based methods.

2. Constitutive law of concrete

Following Desayi and Krishnan [4] and Bergan and Holand [1], the uniaxial stress–strain relation for concrete is given by a function, which is smooth almost everywhere, except at a finite number of discrete points (Fig. 1):

$$\sigma(\varepsilon) = \begin{cases} 0 & \varepsilon \leqslant \varepsilon_{\mathrm{u}}, \\ 2f_{\mathrm{m}}|\varepsilon_1|\dfrac{\varepsilon}{\varepsilon_1^2+\varepsilon^2} & \varepsilon_{\mathrm{u}} < \varepsilon \leqslant \varepsilon_{\mathrm{r}}, \\ \dfrac{\sigma_{\mathrm{r}}}{\varepsilon_{\mathrm{r}}-\varepsilon_{\mathrm{m}}}(\varepsilon-\varepsilon_{\mathrm{m}}) & \varepsilon_{\mathrm{r}} < \varepsilon \leqslant \varepsilon_{\mathrm{m}}, \\ 0 & \varepsilon_{\mathrm{m}} < \varepsilon. \end{cases} \tag{1}$$

Here f_{m} is strength of concrete in compression ($f_{\mathrm{m}} = |\sigma_{\min}| > 0$); $\varepsilon_1 < 0$ is strain at f_{m}; $\varepsilon_{\mathrm{u}} < 0$ is ultimate

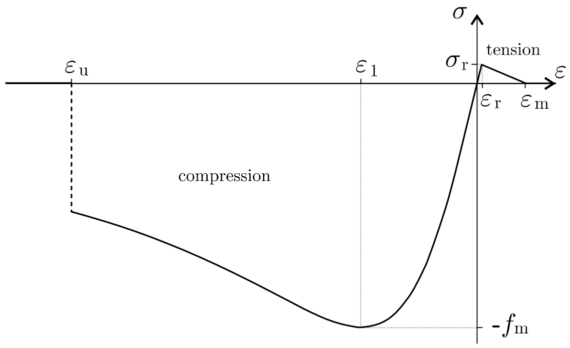

Fig. 1. Constitutive law of concrete.

strain in compression; $\varepsilon_{\mathrm{r}} > 0$ is strain at tension strength of concrete, $\sigma_{\mathrm{r}} = 2f_{\mathrm{m}}|\varepsilon_1|\frac{\varepsilon_{\mathrm{r}}}{\varepsilon_1^2+\varepsilon_{\mathrm{r}}^2}$; and $\varepsilon_{\mathrm{m}} > 0$ is ultimate strain in tension. Parameters f_{m}, ε_1, and ε_{u} are determined in the compression tests on concrete cylinders; ε_{r} and ε_{m} must be determined in tension tests which are for concrete only rarely performed. The empirically proved and commonly used values $\varepsilon_{\mathrm{r}} = 5.5 \times 10^{-5}$ and $\varepsilon_{\mathrm{m}} = 7 \times 10^{-4}$ are rather good approximative values [1].

3. Analytical cross-sectional integration

3.1. Strain distribution over the cross-section

In spatial beam elements we usually assume the Bernoulli hypothesis that a cross-section suffers only rigid rotation during deformation. This implies that the normal strain (axial strain) is linearly distributed over the cross-section:

$$\varepsilon(y,z) = \gamma_1 + y\kappa_3 + z\kappa_2. \tag{2}$$

Here, ε is the normal (axial) strain at fibre (y,z) (see Fig. 2 for the definition of the cross-section and coordinate axes y, z), γ_1 is the normal strain, and κ_2 and κ_3 are the rotational strains (curvatures) about y and z axes,

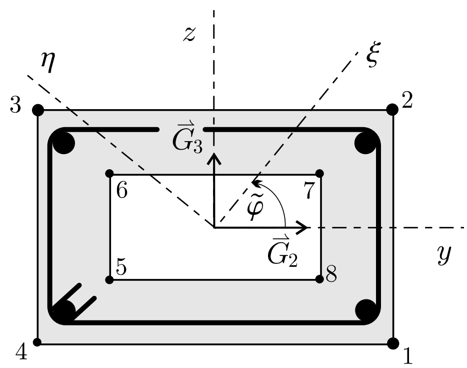

Fig. 2. Model of the cross-section and local coordinate systems.