Engineering Analysis with Boundary Elements 71 (2016) 50–58

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Radial basis function-based pseudospectral method for static analysis of thin plates

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ARTICLE INFO

Article history:
Received 18 November 2015
Accepted 1 July 2016

Keywords:
Radial basis function
Meshless method
Pseudospectral technique
Shape parameter
Kirchhoff plate
Static analysis

ABSTRACT

The paper deals with the implementation of radial basis function-based pseudospectral method (RBF-PS), which is a meshless numerical technique, to static analysis of thin, isotropic plates. The analyzed problem possesses multiple boundary conditions, therefore direct application of the RBF-PS is not straightforward. In the paper, some approaches to implement the method in such the case are presented. They are examined by the analysis of square plates as well as irregular shaped ones with various combinations of boundary conditions. A careful attention is paid to the problem of choosing an appropriate value of the shape parameter for radial basis functions and an effective approach is explained. The obtained results show the usefulness and high accuracy of the approaches proposed, confirming the advantages of the use of the radial basis function approximation in pseudospectral mode.

1. Introduction

Since it was found that radial basis functions (RBFs) are very useful in approximation of scattered points [1], these functions have been extensively applied in numerical methods of solving differential equations. In 1990 Kansa used the RBF interpolation and collocation technique to obtain the solution of differential equations [2,3], proving, by numerical tests, high efficiency of the approach. Since then, many other meshless methods have taken advantage of the RBF approximation. Some of them are based on the weak formulation [4,5], while the others use strong form of the problem considered [6–8]. Among the latter, one can find a method, where RBFs are used in pseudospectral mode. In this technique, called radial basis function-based pseudospectral method (RBF-PS), a global interpolation with RBFs is used to obtain a discrete form of differential operator contained in the analyzed equation and then, with the aid of collocation procedure, the approximate solution. An interesting description of this method can be found in book of Fasshauer [9] and the application of the RBF-PS to some problems from solid mechanics is shown by Ferreira [10], Ferreira and Fasshauer [11,12] and Ferreira et al. [13]. The method has been also applied to electromagnetics problems [14] and to other areas of science [15,16].

It should be noted that similar formulation, named radial basis function-based differential quadrature method, has been introduced by Shu [17] as an extension of the differential quadrature method. The difference lies in the fact that Shu applied the interpolant that uses not only RBFs but also a constant factor.

To protect the global RBF collocation techniques from ill-conditioning problems, some localized versions of the RBF-based methods have been proposed and applied in some areas of engineering, e.g. in fluid flow problems [18,19], heating phenomenon [20,21] and natural convection [22].

The research done so far indicates that the use of RBFs in pseudospectral mode combines advantages of the meshless methods (possibilities of the discretization of the domain using scattered nodes) with high accuracy of the pseudospectral techniques.

In present paper, the RBF-PS is applied to static analysis of thin, isotropic plates. The problem is governed by the fourth order equation with multiple boundary conditions. The direct use of the RBF-PS in this case is troublesome since collocation procedure allows to associate one discrete equation with one node. Therefore the problem with implementation of multiple boundary conditions arises. To overcome this problem, three approaches are shown in the present paper: the first one – leading to overdetermined system of equations, the second one – with simplified implementation of boundary conditions and the third one – based on Hermite type interpolation. The problem of choosing the value of the shape parameter for RBFs is also investigated and an effective approach is presented. Finally, the method is applied to find the deflection of square plates as well as irregular shaped ones.

2. Radial basis function-based pseudospectral methods

Let us consider boundary value problem described in general

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http://dx.doi.org/10.1016/j.enganabound.2016.07.002
0955-7997/

way as

$$Lu = f \quad \text{in } \Omega \tag{1}$$

where L is a differential operator, u and f denote the sought function and the known one, respectively. Assuming that L is a differential operator of higher order, more than one boundary condition is required to have well-posed problem. Without loose of generality let us assume that two boundary conditions are needed to make Eq. (1) well-posed, i.e.

$$B_1 u = g_1,\ B_2 u = g_2 \quad \text{on } \partial\Omega \tag{2}$$

where B_1, B_2 are differential operators acting on the sought function at boundary $\partial\Omega$ and g_1, g_2 denote the functions defined on this boundary.

2.1. Direct RBF-PS approach

In the RBF-PS method the domain of the problem including the boundary is represented by scattered nodes $\mathbf{x}_i$, $i = 1, \dots, N$. Among them one can distinguish interior nodes $\mathbf{x}_i^I$, $i = 1, \dots, N^I$ and the nodes imposed on the boundary $\mathbf{x}_i^B$, $i = 1, \dots, N^B$. Using these nodes, the sought function is interpolated with RBFs as follows

$$u(\mathbf{x}) = \sum_{j=1}^{N} \alpha_j \varphi\left(\|\mathbf{x} - \xi_j\| \right) \tag{3}$$

In Eq. (3) α_j are the interpolation coefficients and $\varphi_j(\mathbf{x}) = \varphi\left(\|\mathbf{x} - \xi_j\| \right)$ are the RBFs, where $\boldsymbol{\xi}_j$ denote the special points called centers. In the approach presented, these special points are taken as the nodes. With each of nodes unknown function value u_i is associated.

There are various types of RBFs [23]. In present paper the multiquadrics RBFs are used since they have been found to be very accurate in scattered data interpolation and are characterized by so-called exponential convergence [1].

Using interpolation conditions

$$\sum_{j=1}^{N} \alpha_j \varphi\left(\|\mathbf{x}_i - \mathbf{x}_j\| \right) = u_i, \quad i = 1, \dots, N \tag{4}$$

one can express interpolation coefficients in terms of the sought function values, what can be put in matrix notation as

$$\boldsymbol{\alpha} = \boldsymbol{\Phi}^{-1}\mathbf{u} \tag{5}$$

where $\boldsymbol{\alpha} = \left[\alpha_1 \ \cdots \ \alpha_N \right]^T$, $\mathbf{u} = \left[u_1 \ \cdots \ u_N \right]^T$ and $\boldsymbol{\Phi}_{ij} = \varphi(\|\mathbf{x}_i - \mathbf{x}_j\|)$, $i, j = 1, \dots, N$.

The next step is to impose differential operator L, contained in Eq. (1), on the interpolation function Eq. (3) and evaluate it at each interior node $\mathbf{x}_i^I$, $i = 1, \dots, N^I$, what yields

$$\mathbf{u}_L = \boldsymbol{\Phi}_L \boldsymbol{\alpha} \tag{6}$$

where $(\mathbf{u}_L)_i = \left[Lu(\mathbf{x}) \right]_{\mathbf{x}=\mathbf{x}_i^I}$, $i = 1, \dots, N^I$, $\left(\boldsymbol{\Phi}_L \right)_{ij} = \left[L\varphi\left(\|\mathbf{x} - \xi\| \right) \right]_{\mathbf{x}=\mathbf{x}_i^I}^{\xi=\mathbf{x}_j}$,

$i = 1, \dots, N^I, j = 1, \dots, N$.

In Eq. (6) L acts on the radial function treated as a function of $\mathbf{x}$ variable.

Taking advantage of Eq. (5) one can express derivative $\mathbf{u}_L$ in Eq. (6) in terms of the sought function values from all over domain as

$$\mathbf{u}_L = \boldsymbol{\Phi}_L \boldsymbol{\Phi}^{-1}\mathbf{u} \tag{7}$$

where $\boldsymbol{\Phi}_L \boldsymbol{\Phi}^{-1}$ is a discrete form of differential operator L and is called as differentiation matrix in the nomenclature of pseudospectral methods.

Similar operation has to be done to approximate the differential operators contained in boundary conditions Eq. (2). In this case the derivatives are evaluated at each of boundary nodes, what yields

$$\mathbf{u}_{B_1} = \boldsymbol{\Phi}_{B_1}\boldsymbol{\Phi}^{-1}\mathbf{u}, \quad \mathbf{u}_{B_2} = \boldsymbol{\Phi}_{B_2}\boldsymbol{\Phi}^{-1}\mathbf{u} \tag{8}$$

where

$\left(\mathbf{u}_{B_1} \right)_i = \left[B_1 u(\mathbf{x}) \right]_{\mathbf{x}=\mathbf{x}_i^B}$, $i = 1, \dots, N^B$, $\left(\boldsymbol{\Phi}_{B_1} \right)_{ij} = \left[B_1 \varphi\left(\|\mathbf{x} - \xi\| \right) \right]_{\mathbf{x}=\mathbf{x}_i^B}^{\xi=\mathbf{x}_j}$,

$i = 1, \dots, N^B, j = 1, \dots, N$

and

$\left(\mathbf{u}_{B_2} \right)_i = \left[B_2 u(\mathbf{x}) \right]_{\mathbf{x}=\mathbf{x}_i^B}$, $i = 1, \dots, N^B$,

$\left(\boldsymbol{\Phi}_{B_2} \right)_{ij} = \left[B_2 \varphi\left(\|\mathbf{x} - \xi\| \right) \right]_{\mathbf{x}=\mathbf{x}_i^B}^{\xi=\mathbf{x}_j}$, $i = 1, \dots, N^B, j = 1, \dots, N$.

With above approximations, Eq. (1) and associated boundary conditions Eq. (2) can be expressed in the form of algebraic set of equations as

$$\boldsymbol{\Phi}_{LB}\boldsymbol{\Phi}^{-1}\mathbf{u} = \mathbf{q} \tag{9}$$

where $\boldsymbol{\Phi}_{LB} = \left[\boldsymbol{\Phi}_L \ \ \boldsymbol{\Phi}_{B_1} \ \ \boldsymbol{\Phi}_{B_2} \right]^T$ is the $\left(N^I + 2N^B \right)$ by N matrix, $\boldsymbol{\Phi}_{LB}\boldsymbol{\Phi}^{-1}$ can be considered as an extension of the differentiation matrix, which approximates operator L at interior nodes and operators B_1, B_2 at boundary nodes and $\mathbf{q} = \left[\mathbf{f} \ \ \mathbf{g}_1 \ \ \mathbf{g}_2 \right]^T$ is $N^I + 2N^B$ element vector containing values of function f at interior nodes and values of functions g_1, g_2 at boundary nodes.

One can notice from Eq. (9) that direct use of RBF-PS method in the case of boundary value problem possessing multiple boundary conditions leads to overdetermined set of equations.

To obtain the approximate solution of Eq. (9) least squares technique can be employed, what yields

$$\mathbf{u} = \left(\mathbf{A}^T\mathbf{A} \right)^{-1} \cdot \mathbf{A}^T \cdot \mathbf{q} \tag{10}$$

where $\mathbf{A} = \boldsymbol{\Phi}_{LB}\boldsymbol{\Phi}^{-1}$.

2.2. Direct RBF-PS approach with simplified implementation of boundary conditions

To overcome the inconvenience following from overdetermined set of equations but to maintain the simplicity of the method, another approach is proposed.

In this approach, the boundary nodes $\mathbf{x}_j^B$, $j = 1, \dots, N^B$ are split into appropriate number of groups. This number corresponds to the number of boundary conditions. The nodes are chosen by turns to appropriate group. Each boundary condition is implemented at nodes from another group. For example, in the case of two boundary conditions, the set of boundary nodes are split into two groups of nodes $\mathbf{x}_j^{B_1}$, $j = 1, \dots, N^{B_1}$ and $\mathbf{x}_j^{B_2}$, $j = 1, \dots, N^{B_2}$. Note that $N^{B_1} + N^{B_2} = N^B$ is the number of all nodes imposed on boundaries of the domain. One boundary condition is implemented at $\mathbf{x}_j^{B_1}$, $j = 1, \dots, N^{B_1}$, while the other at $\mathbf{x}_j^{B_2}$, $j = 1, \dots, N^{B_2}$, what is shown in Fig. 1.

To derive the differentiation matrix one has to modify interpolant Eq. (3), introducing terms associated with boundary conditions, evaluated at appropriate nodes. In this case the

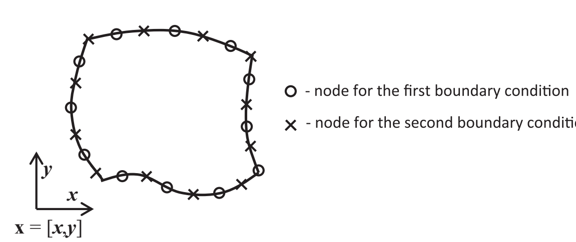

Fig. 1. Idea of the implementation of boundary conditions.