Engineering Analysis with Boundary Elements 71 (2016) 140–150

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Applications of the Clifford algebra valued boundary element method to electromagnetic scattering problems

Jia-Wei Lee [a], Li-Wei Liu [b], Hong-Ki Hong [b], Jeng-Tzong Chen [a,c,*]

[a] Department of Harbor and River Engineering, National Taiwan Ocean University, Keelung, Taiwan
[b] Department of Civil Engineering, National Taiwan University, Taipei, Taiwan
[c] Department of Mechanical and Mechatronic Engineering, National Taiwan Ocean University, Keelung, Taiwan

ARTICLE INFO

Article history:
Received 29 April 2016
Received in revised form
4 July 2016
Accepted 6 July 2016

Keywords:
Clifford algebra
Clifford algebra valued boundary integral equation
k-Dirac equation
Cauchy-type kernels
Dirac matrices
Electromagnetic scattering

ABSTRACT

Electromagnetic problems governed by Maxwell's equations are solved by using a Clifford algebra valued boundary element method (BEM). The well-known Maxwell's equations consist of eight pieces of scalar partial differential equations of the first order. They can be rewritten in terms of the language of Clifford analysis as a nonhomogeneous k-Dirac equation with a Clifford algebra valued function. It includes three-component electric fields and three-component magnetic fields. Furthermore, we derive Clifford algebra valued boundary integral equations (BIEs) with Cauchy-type kernels and then develop a Clifford algebra valued BEM to solve electromagnetic scattering problems. To deal with the problem of the Cauchy principal value, we use a simple Clifford algebra valued k-monogenic function to exactly evaluate the Cauchy principal value. Free of calculating the solid angle for the boundary point is gained. The remaining boundary integral is easily calculated by using a numerical quadrature except the part of Cauchy principal value. This idea can also preserve the flexibility of numerical method, hence it is suitable for any geometry shape. In the numerical implementation, we introduce an oriented surface element instead of the unit outward normal vector and the ordinary surface element. In addition, we adopt the Dirac matrices to express the bases of Clifford algebra $Cl_3(\mathbb{C})$. We also use an orthogonal matrix to transform global boundary densities into local boundary densities for satisfying boundary condition straightforward. Finally, two electromagnetic scattering problems with a perfect spherical conductor and a prolate spheroidal conductor are both considered to examine the validity of the Clifford algebra valued BEM with Cauchy-type kernels.

1. Introduction

Complex algebra and complex analysis are powerful tools for solving problems in two-dimensional spaces. This point motivates scholars to develop methods based on complex algebra/analysis to deal with problems in three-dimensional spaces. However, due to the algebraic structures of complex numbers and the calculus of complex variables, complex algebra/analysis are improper to be the foundation to naturally develop any numerical method for solving problems in three-dimensional spaces. To preserve some benefits of complex algebra or complex analysis for solving three-dimensional problems, an extension of complex algebra is required.

In 1843, Hamilton [1] proposed quaternion algebra which is an extension of complex algebra in three-dimensional space. Therefore, complex algebra can be seen as a subalgebra of quaternion algebra. Although the multiplication of two quaternion numbers is non-commutative, quaternion algebra can be applied to describe three-dimensional problems. For quaternionic analysis, Fueter and his collaborators started developing it since 1930 [2–5].

On the other hand, Clifford [6] proposed the algebras named after him in 1878. Clifford algebra can be seen as an extension of complex or quaternionic algebras. Bases of Clifford algebra are generated according to the Clifford product rule. In this way, Clifford algebra is different from other algebraic systems that it has no more and no fewer bases to describe the geometric relations of space [7,8] and easier to extend it to higher dimensional problems. As quoted by *Hestenes*: 「*Geometry without algebra is dumb! - Algebra without geometry is blind!*」 [8]. A linear combination of those bases is called Clifford number or multivector. Later, Hestenes [7,9] considered that Clifford algebra can be a common language in physics and mathematics. Clifford algebra has been applied to many fields such as, geometry, dynamics, physics, electromagnetics and information theory [10,11].

As well as the complex analysis in the complex variables and quaternionic analysis [12–14] in the quaternion algebra, there is a new field, Clifford analysis or so-called hypercomplex analysis [15–

* Corresponding author at: Department of Harbor and River Engineering, National Taiwan Ocean University, Keelung, Taiwan.
E-mail addresses: 29952008@mail.ntou.edu.tw, handsome6361@gmail.com (J.-W. Lee), liweiliu@ntu.edu.tw (L.-W. Liu), hkhong@ntu.edu.tw (H.-K. Hong), jtchen@mail.ntou.edu.tw (J.-T. Chen).

http://dx.doi.org/10.1016/j.enganabound.2016.07.007
0955-7997/

20] in Clifford algebra. Liu and Hong used Clifford analysis to derive general solutions of both isotropic elasticity [21] and anisotropic elasticity [22,23]. One goal of this paper is to develop the Clifford algebra valued BIE with the Cauchy-type kernel to deal with three-dimensional problems. Regarding Clifford algebra valued BIE [24], Hong and Liu derived it [25,26] and employed it to solve three-dimensional magnetostatic problems [27] and three-dimensional elasticity [28]. However, they merely focused on static problems.

Now, we extend Clifford algebra valued BIE to solve three-dimensional time-harmonic problems. For time-harmonic problems, such as Helmholtz equations, Gerus and Shapiro [29] derived a Cauchy-type integral corresponding to the two-dimensional Helmholtz equation by using quaternion algebra. Later, Vu Thi Ngoc Ha and Begehr [30] extended to the three-dimensional Helmholtz equation. Both works focused on deriving the Cauchy-type integral, however no numerical results were provided. Chantaveerod and his coworkers [31–33] employed the four-dimensional Clifford algebra to describe Maxwell's equations as a k-Dirac equation. They called it Maxwell-Dirac Equation. They also employed the corresponding Clifford algebra valued BIE to calculate both interior and exterior problems with a plane wave and a Hertzian dipole, respectively.

In this paper, we employ the three-dimensional Clifford algebra $Cl_3(\mathbb{C})$ and Clifford analysis to reformulate Maxwell's equations to a k-Dirac equation. We believe that algebraic space of the three-dimensional Clifford algebra is sufficient to describe Maxwell's equations. Furthermore, we derive Clifford algebra valued BIEs for the k-Dirac equation and develop its Clifford algebra valued BEM. To examine the validity of Clifford algebra valued BEM with the Cauchy-type kernel, two electromagnetic scattering problem with a perfect conductor and a prolate spheroidal conductor are considered. Finally, the numerical results obtained from Clifford algebra valued BEM show a good agreement with those of finite element method (FEM) [34] and method of fundamental solutions (MFS) [35].

2. Problem statement of electromagnetic scattering

The typical electromagnetic scattering problem in the frequency domain is governed by Maxwell's equations as shown below:

$$\nabla\cdot\vec{D}(x) = \rho_f(x), \tag{1}$$

$$\nabla\times\vec{H}(x) - i\omega\vec{D}(x) = \vec{J}_f(x), \tag{2}$$

$$\nabla\times\vec{E}(x) + i\omega\vec{B}(x) = 0, \tag{3}$$

$$\nabla\cdot\vec{B}(x) = 0, \tag{4}$$

where $\vec{E}(x)$ is the electric field, $\vec{D}(x)$ is the electric displacement field, $\vec{B}(x)$ is the magnetic flux density, $\vec{H}(x)$ is the magnetic field strength, $\vec{J}_f(x)$ is the free current density, $\rho_f(x)$ is the free charge density and ω is the angular frequency. For the linear isotropic medium, the constitutive equations are shown below:

$$\vec{D}(x) = \varepsilon\vec{E}(x), \tag{5}$$

$$\vec{H}(x) = \frac{1}{\mu}\vec{B}(x), \tag{6}$$

where ε and μ are the permittivity and the permeability, respectively. The relation between ε and μ is

$$\frac{1}{\sqrt{\varepsilon\mu}} = \frac{\omega}{k} = c, \tag{7}$$

where k and c denote the wave number and the speed of the electromagnetic wave. In a vacuum, it is usually to use the symbols ε_0 and μ_0 to stand for the permittivity and the permeability, respectively. The values of ε_0 and μ_0 are

$$\varepsilon_0 \approx 8.854187817\times10^{-12}\ \mathrm{A^2{\cdot}s^4{\cdot}kg^{-1}{\cdot}m^{-3}}, \tag{8}$$

$$\mu_0 = 4\pi\times10^{-7}\ \mathrm{A{\cdot}s^{-2}{\cdot}kg}{\cdot}m, \tag{9}$$

respectively. Also, c_0 is the speed of light in a vacuum and its value is

$$c_0 = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = 299792458\ \mathrm{m{\cdot}s^{-1}}. \tag{10}$$

Substituting Eqs. (5) to (7) into Eqs. (1) to (4), we obtain

$$\nabla\cdot\vec{E}(x) = \frac{\rho_f(x)}{\varepsilon}, \tag{11}$$

$$\nabla\times\frac{\vec{B}(x)}{\sqrt{\varepsilon\mu}} - ik\vec{E}(x) = \sqrt{\frac{\mu}{\varepsilon}}\vec{J}_f(x), \tag{12}$$

$$\nabla\times\vec{E}(x) + ik\frac{\vec{B}(x)}{\sqrt{\varepsilon\mu}} = 0, \tag{13}$$

$$\nabla\cdot\vec{B}(x) = 0. \tag{14}$$

The boundary conditions for an electromagnetic scattering problem with a perfect conductor are

$$\vec{n}(x)\times\vec{E}(x) = 0, \tag{15}$$

$$\vec{n}(x)\cdot\vec{B}(x) = 0, \tag{16}$$

where $\vec{n}(x)$ is a unit inward normal vector on the conductor surface and

$$\vec{E}(x) = \vec{E}^{in}(x) + \vec{E}^{s}(x),\quad \vec{E}^{(\cdot)}(x) = (E_1^{(\cdot)}(x), E_2^{(\cdot)}(x), E_3^{(\cdot)}(x)), \tag{17}$$

$$\vec{B}(x) = \vec{B}^{in}(x) + \vec{B}^{s}(x),\quad \vec{B}^{(\cdot)}(x) = (B_1^{(\cdot)}(x), B_2^{(\cdot)}(x), B_3^{(\cdot)}(x)), \tag{18}$$

in which the superscripts "*in*" and "*s*" stand for the incident wave and the scattering field, respectively.

3. Clifford algebra and Clifford analysis in $Cl_3(\mathbb{C})$

3.1. Algebraic structures of Clifford algebra $Cl_3(\mathbb{C})$

The Clifford product e_je_k (in that order, denoted by juxtaposition) of e_j and e_k is defined by the Clifford product rule

$$e_je_k + e_ke_j = 2\delta_{jk},\ j, k = 1, 2, 3. \tag{19}$$

The basis elements of Clifford algebra in three-dimensional Euclidean space are generated from $e_\varnothing = 1$ and $\{e_j\}$ by the rule of Eq. (19) and we have eight bases