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Single layer regularized meshless method for three dimensional Laplace problem

Lin Liu *, Hong Zhang

Department of Propulsion & Fuel system, Shanghai Aircraft Development and Research Institute, Shanghai 201210, PR China

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ABSTRACT

The subtraction and adding-back technique for Regularized Meshless Method (RMM) has been proposed by Young et al. (2005) [8] on 2-D Laplace problem and extended to 3-D Laplace problem by Young et al. (2009) [13], where the kernel functions of double layer potentials were adopted to desingularize fundamental solution singularity while the source points are overlapped on the physical points. Here the Single Layer Regularized Meshless Method (SRMM) is proposed. The solutions are represented by single layer potential. The singularity of the fundamental solution is desingularized by the carefully chosen particular solution in the null-fields of the boundary integral equation using the subtraction and adding-back technique for the Dirichlet boundary condition. The double layer potential is adopted for the Neumann boundary condition. The numerical examples show that the convergence trend and accuracy of the SRMM are better than those of using other methods (RMM, IBDS) by one or two orders of magnitude.

1. Introduction

The Method of Fundamental Solutions (MFS) is a typical meshless boundary collocation method. However the choice of source points is arbitrary and without a particular rule. Many Boundary Meshless Methods with source points coincident with physical points have been proposed in the literature. These methods use different techniques to avoid the singularity of fundamental solution. Boundary Node Method (BNM) [1] adopts the interpolation procedure to circumvent the singularity. Boundary Points Method (BPM) [2] uses the 'moving elements' to avoid the singularity. Boundary Particle Method (BPM) [3], Boundary Knot Method (BKM) [4] employ an alternative non-singular kernel function to circumvent the singularity. Boundary Distributed Source (BDS) method [5], Improved Boundary Distributed Source (IBDS) [6], Non-Singular Method of fundamental solution [7] removes the singularities by distributed source over areas (for 2D) or volumes (for 3D) covering the source points.

Regularized Meshless Method (RMM) which uses the desingularization of subtracting and adding back technique was proposed by Young et al. (2005) [8] for 2-D Laplace problem, and then applied to different problems [9–12]. This method was later extended to 3-D Laplace problem [13]. The double layer potential was adopted as the fundamental solution for the convenience of using null-fields boundary integral equation to desingularize the fundamental solution. For the Dirichlet problem, the particular solution $u(s) = 1$ was used in the null-fields boundary integral equation to get the diagonal elements. In this paper the particular solution $\boldsymbol{p} = x + y + z$, $u(s) = \langle \boldsymbol{n}_x, (\boldsymbol{p}_s - \boldsymbol{p}_x) \rangle$ is chosen for the null-fields boundary integral equation to derive the diagonal elements for the Dirichlet problem. By this particular solution, the diagonal elements can be represented by the single layer potential.

This paper is also similar to the idea of Singular Boundary Method [14,15] to get the magnitude of singular source, which also uses the single layer as the fundamental solution. But in the SBM [14] or ISBM [15], the inverse interpolation technique (IIT) was adopted to get the singular source magnitude for the Dirichlet problem. They got the Neumann problem singular source magnitude by the subtraction and adding-back technique in null-fields boundary integral equation firstly, then integrated the solution to achieve the Dirichlet problem singular source magnitude, the constant for the integration was derived by the inverse interpolation from the domain points. In this paper, the Dirichlet problem singular source magnitude is directly derived from the null-field integral equation by the subtraction and adding-back technique using the single layer potential, without the need of inverse interpolation.

In the following sections, the theory of the Single Layer Regularized Meshless Method (SRMM) is introduced. Then the Dirichlet problem and Neumann problem for sphere, ring torus, peanut are tested by the proposed SRMM, and also compared with the RMM [13] and IBDS [7].

* Corresponding author.
E-mail address: liulin@comac.cc (L. Liu).

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2. Formulation of single layer regularized meshless method

From the boundary value problem Laplace equation in 3D domain Ω

$$\nabla^2 u(x) = 0, \quad x \in \Omega \tag{1}$$

subject to the following boundary conditions:

$$u(x) = \bar{u}(x), \quad x \in \Gamma_D \quad \text{(Dirichlet boundary condition)} \tag{2}$$

$$q(x) = \frac{\partial u}{\partial \boldsymbol{n}}(x) = \bar{q}(x), \quad x \in \Gamma_N \quad \text{(Neumann boundary condition)} \tag{3}$$

where u is the potential field, Ω is a bounded domain with boundary $\Gamma = \Gamma_D + \Gamma_N$, $\boldsymbol{n}$ presents the outward normal unit vector at x.

The solution $u(x)$ and $\partial u(x)/\partial \boldsymbol{n}$ of the Laplace problem can be approximated by a linear combination of the fundamental solution with respect to different source points s_j as follows:

$$u(x_i) = \begin{cases} \sum_{j=1,}^{N} \alpha_j G(x_i, s_j), & x_i \in \Omega \\ \sum_{j=1, j\neq i}^{N} \alpha_j G(x_i, s_j) + \alpha_i U(x_i, s_i), & x_i \in \Gamma \end{cases} \tag{4}$$

$$q(x_i) = \frac{\partial u(x_i)}{\partial n_{x_i}} = \begin{cases} \sum_{j=1}^{N} \alpha_j \frac{\partial G(x_i, s_j)}{\partial \boldsymbol{n}_{x_i}}, & x_i \in \Omega \\ \sum_{j=1, j\neq i}^{N} \alpha_j \frac{\partial G(x_i, s_j)}{\partial \boldsymbol{n}_{x_i}} + \alpha_i Q(x_i, s_i), & x_i \in \Gamma \end{cases} \tag{5}$$

where x_i is the ith physical point, s_j is the jth source point located on the physical boundary, α_j is the jth unknown intensity of the distributed source at s_j, N is the number of source points and

$$G(x_i, s_j) = -\frac{1}{r}, \tag{6}$$

$$\frac{\partial G(x_i, s_j)}{\partial \boldsymbol{n}_{x_i}} = \frac{\langle (x_i - s_j), \boldsymbol{n}_{x_i} \rangle}{r^3} \tag{7}$$

are the fundamental solution and its physical normal derivation of three-dimensional Laplace equation, $r = \| x_i - s_j \|$, $\langle \, , \rangle$ denotes the inner product. In Eqs. (4) and (5) the α_j can be solved by the linear equations which satisfy the boundary condition. The only unknowns are the $U(x_i, s_i)$ for the Dirichlet problem and $Q(x_i, s_i)$ for the Neumann problem. Followed by Young et al. (2005) [8], the null-fields of the boundary integral equation on the direct method is used to derive the $Q(x_i, s_i)$.

$$0 = \int_\Gamma \left(u(s) \frac{\partial G(x, s)}{\partial \boldsymbol{n}_s^e} - G(x, s) \frac{\partial u(s)}{\partial \boldsymbol{n}_s^e} \right) d\Gamma(s), \quad x \in \Omega^e \tag{8}$$

where the superscript e denotes the exterior domain. If the particular solution $u(s) = 1$ and $\partial u(s)/\partial \boldsymbol{n}_s^e = 0$ are chosen, Eq. (8) can be rewritten as follows:

$$\int_\Gamma \frac{\partial G(x, s)}{\partial \boldsymbol{n}_s^e} d\Gamma(s) = 0, \quad x \in \Omega^e \tag{9}$$

When the collocation point x approaches the boundary, we can discretize Eq. (9) as follows:

$$\sum_{j=1}^{N} \frac{\partial G(x_i, s_j)}{\partial \boldsymbol{n}_{s_j}^e} S_j = 0, \quad x \in \Gamma \tag{10}$$

where S_j is the element area around s_j, $\frac{\partial G(x_i, s_j)}{\partial \boldsymbol{n}_{s_j}} = -\frac{\langle (x_i - s_j), \boldsymbol{n}_{s_j} \rangle}{r^3}$. When the source point s_j moves close to the collocation point x_i, we have

$$\lim_{s_j \to x_i} \frac{\partial G(x_i, s_j)}{\partial \boldsymbol{n}_{x_i}^e} + \frac{\partial G(x_i, s_j)}{\partial \boldsymbol{n}_{s_j}^e} = 0 \tag{11}$$

So we get $Q^e(x_i, s_i)$

$$Q^e(x_i, s_i) = \frac{1}{S_i} \sum_{j=1, j\neq i}^{N} S_j \frac{\partial G(x_i, s_j)}{\partial \boldsymbol{n}_{s_j}^e} \tag{12}$$

If the nodes on the boundary are uniformly distributed, then

$$Q^e(x_i, s_i) = \sum_{j=1, j\neq i}^{N} \frac{\partial G(x_i, s_j)}{\partial \boldsymbol{n}_{s_j}^e} \tag{13}$$

Now we choose other particular solution $u(s)$ for the Laplace equation to derive $U^e(x_i, s_i)$.

$$\boldsymbol{p} := x + y + z \tag{14}$$

$$u(s) = \langle \boldsymbol{n}_x, (\boldsymbol{p}_s - \boldsymbol{p}_x) \rangle \tag{15}$$

$$\frac{\partial u(s)}{\partial \boldsymbol{n}_s} = \langle \nabla u(s), \boldsymbol{n}_s \rangle = \langle \boldsymbol{n}_x, \boldsymbol{n}_s \rangle \tag{16}$$

Then the null-fields equation (8)

$$\int_\Gamma \left(\langle \boldsymbol{n}_x^e, (\boldsymbol{p}_s - \boldsymbol{p}_x) \rangle \frac{\partial G(x, s)}{\partial \boldsymbol{n}_s^e} - G(x, s) \langle \boldsymbol{n}_x^e, \boldsymbol{n}_s^e \rangle \right) d\Gamma(s) = 0, \quad x \in \Omega^e \tag{17}$$

When the collocation point x approaches the boundary, we can discretize Eq. (17) as follows:

$$U^e(x_i, s_i) = \frac{1}{S_i} \sum_{j=1, j\neq i}^{N} S_j \left\{ \frac{\partial G(x_i, s_j)}{\partial \boldsymbol{n}_{s_j}^e} \langle \boldsymbol{n}_{x_i}^e, (\boldsymbol{p}_{s_j} - \boldsymbol{p}_{x_i}) \rangle - G(x_i, s_j) \langle \boldsymbol{n}_{x_i}^e, \boldsymbol{n}_{s_j}^e \rangle \right\} \tag{18}$$

If the nodes on the boundary are uniformly distributed, then

$$U^e(x_i, s_i) = \sum_{j=1, j\neq i}^{N} \left\{ \frac{\partial G(x_i, s_j)}{\partial \boldsymbol{n}_{s_j}^e} \langle \boldsymbol{n}_{x_i}^e, (\boldsymbol{p}_{s_j} - \boldsymbol{p}_{x_i}) \rangle - G(x_i, s_j) \langle \boldsymbol{n}_{x_i}^e, \boldsymbol{n}_{s_j}^e \rangle \right\} \tag{19}$$

The benefits of using the particular solution Eq. (15) to derive Eq. (18) are obvious: when the $x_i \to s_j$, on the one hand the coefficient for $U^e(x_i, s_i)$ is 1; on the other hand, the coefficient for $\frac{\partial G(x_i, s_i)}{\partial \boldsymbol{n}_{s_i}^e}$ is 0, which is avoided to solve the singularity of $\frac{\partial G(x_i, s_i)}{\partial \boldsymbol{n}_{s_i}^e}$. If another particular solution $u(s)$ is selected for the Laplace equation to derive $U^e(x_i, s_i)$, these benefits disappear and the error for $U^e(x_i, s_i)$ increase.

According to the dependency of the normal vectors on the fundamental solutions of interior and exterior Laplace equation,

$$Q^i(x_i, s_i) = \frac{\partial G(x_i, s_i)}{\partial \boldsymbol{n}_{x_i}^i} = -\frac{\partial G(x_i, s_i)}{\partial \boldsymbol{n}_{x_i}^e} = -Q^e(x_i, s_i) \tag{20}$$

$$U^i(x_i, s_i) = U^e(x_i, s_i) \tag{21}$$

where the superscript i denotes the interior domain.