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Interior field methods for Neumann problems of Laplace's equation in elliptic domains, comparisons with degenerate scales ☆

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ABSTRACT

The interior field method (IFM) is applied to Neumann problems for Laplace's equation in elliptic domains. The polynomial convergence rates are derived, and small condition number as $O(N)$ can be obtained, where N is the number of particular solutions used. Moreover, the effective condition number as $O(1)$ is explored, to display excellent stability (Li et al., 2015 [21]). Numerical experiments are carried out, to support the analysis made. The error analysis of the IFM for Dirichlet problems in circular domains is reported in Li et al. (2016) [19]. The error and stability analysis of the IFM for Neumann problems in elliptic domains is more advanced and challenging; this is the first goal of this paper. The second goal is to compare the Neumann problems with the degenerate scales of Dirichlet problems; some useful guidances are found for application. From the comparisons, the conservation law is essential to guarantee the unique solutions. The adaptive processes are also proposed to deal with the algorithm singularity of Dirichlet problems; they may be applied to the boundary element method (BEM), the original NFM, and the indirect BIEM for the arbitrary smooth boundary or the convex polygonal boundary.

1. Introduction

The interior field method (IFM) and the null field method (NFM) have been developed in [23,25] for Laplace's equation in elliptic domains, where the explicit algebraic equations are derived. Only the Dirichlet problems and the mixed problems are studied in [23,25], but the Neumann problems are the theme of this paper. A strict analysis of errors and stability is important for any numerical method. The error analysis of the IFM is provided in [19] for Dirichlet problems of Laplace's equation in circular domains. However, the error and stability analysis of the IFM for Neumann problems in elliptic domains is more advanced and challenging; this is the first goal of this paper. The polynomial convergence rates are derived, and small condition number as $O(N)$ can be obtained, where N is the number of particular solutions used. Moreover, the effective condition number as $O(1)$ is explored, to display excellent stability [21]. Numerical experiments are carried out, to support the analysis made. Since the interior field method (IFM) is equivalent to the second kind NFM, when the field nodes Q are located on the domain boundary, the error and stability analysis in this paper is also valid for the second kind NFM when $Q \in \partial S$ (see [16,25]).

The second goal is to compare Neumann problems with degenerate scales of Dirichlet problems; some useful guidances are found for application. The conservation law of flux is essential for seeking the unique solutions in degenerate scales. Moreover, the adaptive processes are proposed, to deal with the algorithm singularity of the IFM and the NFM for Dirichlet problems. The adaptive processes may be applied to the BEM, the original NFM, and the indirect BIEM for arbitrary smooth boundary and the convex polygonal boundary.

More references related to the IFM and the NFM are given in [19]. This paper is organized as follows. In the next section, the algorithms of the IFM are described for Neumann problems in elliptic domains with one elliptic hole. In Section 3, the error bounds are derived for the IFM in Sobolev norms, to prove the polynomial convergence rates. In Section 4, the stability analysis is explored for the collocation IFM (CIFM), and in Section 5, numerical experiments are carried out. In Section 6, comparisons are made with the degenerate scales of Dirichlet problems, and the adaptive processes are proposed. In the last section, a few

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concluding remarks are made.

2. Numerical algorithms

Consider the Neumann problem in an annular domain S (see Fig. 1),

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ in } S, \tag{2.1}$$

$$u_\nu = f \text{ on } \partial S_R, \quad u_\nu = g \text{ on } \partial S_{R_1}, \tag{2.2}$$

where f and g are the known functions, ∂S_R and ∂S_{R_1} are the exterior and the interior elliptic boundaries of S, respectively, and $u_\nu = \frac{\partial u}{\partial \nu}$ is the exterior normal derivatives on $\partial S_R \cup \partial S_{R_1}$. For the existence of any solution, the exterior and the interior boundaries are assumed to have no overlap (see [19]),

$$\partial S_R \cap \partial S_{R_1} = \emptyset. \tag{2.3}$$

Also the known functions in (2.2) must obey the consistent condition (i.e., the conservation law of flux),

$$\iint_S \Delta u = \int_{\partial S_R} \frac{\partial u}{\partial \nu} + \int_{\partial S_{R_1}} \frac{\partial u}{\partial \nu} = 0, \tag{2.4}$$

to guarantee the existence of infinite solutions. Hence, a solution of (2.1) and (2.2) plus any constant is also the solution. For elliptic domains, we need the elliptic coordinates (ρ, θ) defined by

$$x = \sigma_0 \cosh \rho \cos \theta, \quad y = \sigma_0 \sinh \rho \sin \theta, \tag{2.5}$$

where $\sigma_0 > 0, 0 \le \rho < \infty$ and $0 \le \theta \le 2\pi$. When $\rho = \rho_0 > 0$, Eq. (2.5) leads to an ellipse:

$$\frac{x^2}{\sigma_0^2 \cosh^2 \rho_0} + \frac{y^2}{\sigma_0^2 \sinh^2 \rho_0} = 1, \tag{2.6}$$

where two semi-axes are $a = \sigma_0 \cosh \rho_0$ and $b = \sigma_0 \sinh \rho_0$. Let the exterior ellipse S_R by $\rho \le R$ be located with origin $(0, 0)$, where the elliptic coordinates (ρ, θ) are given in (2.5). The local elliptic coordinates $(\bar\rho, \bar\theta)$ are defined by

$$\bar x = \sigma_1 \cosh \bar\rho \cos \bar\theta, \quad \bar y = \sigma_1 \sinh \bar\rho \sin \bar\theta, \quad \sigma_1 > 0, \tag{2.7}$$

with origin (x_1, y_1). The Cartesian coordinates $(\bar x, \bar y)$ in (2.7) are rotated from axis X, by a counter-clockwise angle Θ as shown in Fig. 1. Then the interior ellipse S_{R_1} is denoted by $\bar\rho \le R_1$ in coordinates $(\bar\rho, \bar\theta)$. The annular domain denoted by $S = S_R \backslash S_{R_1}$, and its boundary by $\partial S = \partial S_R \cup \partial S_{R_1}$.

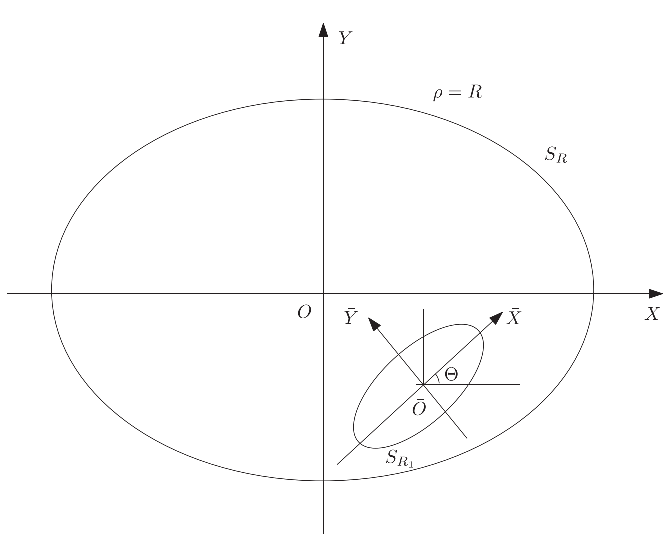

Fig. 1. The annular domain of ellipse S_R with an elliptic hole S_{R_1}.

The functions f and g in (2.2) are assumed to have approximations of series from [23],

$$\frac{\partial u}{\partial \nu} \approx \frac{1}{\sigma_0 \tau_0(\theta)} \left\{ p_0 + \sum_{k=1}^{M} \{p_k \cos k\theta + q_k \sin k\theta\} \right\} \quad \text{on } \partial S_R, \tag{2.8}$$

$$\frac{\partial \bar u}{\partial \bar\nu} = -\frac{\partial \bar u}{\partial \bar\rho} \approx \frac{1}{\sigma_1 \tau_1(\bar\theta)} \left\{ \bar p_0 + \sum_{k=1}^{N} \{\bar p_k \cos k\bar\theta + \bar q_k \sin k\bar\theta\} \right\} \quad \text{on } \partial S_{R_1}, \tag{2.9}$$

where $p_k, q_k, \bar p_k$ and $\bar q_k$ are the known coefficients. In (2.8) and (2.9), the functions, $\tau_0(\theta) = \tau_0(R, \theta) = \sqrt{\sinh^2 R + \sin^2 \theta}$ and $\tau_1(\bar\theta) = \tau_1(R_1, \bar\theta) = \sqrt{\sinh^2 R_1 + \sin^2 \bar\theta}$. Substituting (2.8) and (2.9) into conservation law (2.4), the constraint between coefficients p_0 and $\bar p_0$ can be obtained,

$$p_0 + \bar p_0 = 0. \tag{2.10}$$

Suppose that the solutions on ∂S can also be approximated as

$$u \approx a_0 + \sum_{k=1}^{M} \{a_k \cos k\theta + b_k \sin k\theta\} \quad \text{on } \partial S_R, \tag{2.11}$$

$$\bar u \approx \bar a_0 + \sum_{k=1}^{N} \{\bar a_k \cos k\bar\theta + \bar b_k \sin k\bar\theta\} \quad \text{on } \partial S_{R_1}, \tag{2.12}$$

where $a_k, b_k, \bar a_k$ and $\bar b_k$ are unknown coefficients. The interior solutions can be found from [23]

$$\begin{aligned} u_{M-N} &= u_{M-N}(\rho, \theta; \bar\rho, \bar\theta) \\ &= a_0 - \left[R + \ln\left(\frac{\sigma_0}{2}\right)\right] p_0 - \left[\bar\rho + \ln\left(\frac{\sigma_1}{2}\right)\right] \bar p_0 \\ &+ \sum_{k=1}^{M} e^{-kR} \{a_k \cosh k\rho \cos k\theta + b_k \sinh k\rho \sin k\theta\} \\ &+ \sum_{k=1}^{M} \frac{1}{k} e^{-kR} \{p_k \cosh k\rho \cos k\theta + q_k \sinh k\rho \sin k\theta\} \\ &+ \sum_{k=1}^{N} e^{-k\bar\rho} \{\bar a_k \sinh kR_1 \cos k\bar\theta + \bar b_k \cosh kR_1 \sin k\bar\theta\} \\ &+ \sum_{k=1}^{N} \frac{1}{k} e^{-k\bar\rho} \{\bar p_k \cosh kR_1 \cos k\bar\theta + \bar q_k \sinh kR_1 \sin k\bar\theta\}, \quad \text{in } S, \end{aligned} \tag{2.13}$$

where the transformations between (ρ, θ) and $(\bar\rho, \bar\theta)$ are given in [23]. For the Neumann problem, a_0 ($= c$) is an arbitrary constant (also see [16]). In computation, this arbitrary constant is excluded as an unknown coefficient. Note that in (2.13), coefficient $\bar a_0$ is absent, therefore, there are only $(2M + 2N)$ unknown coefficients $a_k, b_k, \bar a_k$ and $\bar b_k$ $(k > 0)$ in (2.13). The interior solutions (2.13) are derived in [23] from the interior field equations under the field nodes inside of S. When $u \in H^3(\partial S_R \cup \partial S_{R_1})$ and $u_\nu \in H^2(\partial S_R \cup \partial S_{R_1})$, the interior solutions (2.13) and their derivatives hold until $\partial S_R \cup \partial S_{R_1}$, based on the analysis in [20,23,25]. Therefore, the unknown coefficients can be sought directly from the Neumann boundary conditions in (2.2) with (2.8) and (2.9):

$$\begin{aligned} &\frac{\partial}{\partial \nu} u_{M-N}(R, \theta; \bar\rho, \bar\theta) \\ &= \frac{1}{\sigma_0 \tau_0(\theta)} \left\{ p_0 + \sum_{k=1}^{M} \{p_k \cos k\theta + q_k \sin k\theta\} \right\} \quad \text{on } \partial S_R, \end{aligned} \tag{2.14}$$