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Free and forced vibration analyses using the four-node quadrilateral element with continuous nodal stress

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ABSTRACT

The recently published four-node quadrilateral element with continuous nodal stress (Quad4-CNS) is extended to free and forced vibration analyses of two-dimensional solids. The Quad4-CNS element can be regarded as a partition-of-unity (PU) based 'FE-Meshfree' element which inherits better accuracy, higher convergence rate, and high tolerance to mesh distortion from the meshfree methods, while preserving the Kronecker-delta property of the finite element method (FEM). Moreover, the Quad4-CNS element is free from the linear dependence problem which otherwise cripples many of the PU based finite elements. Several free and forced vibration problems are solved and the performance of the element is compared with that of the four-node isoparametric quadrilateral element (Quad4) and eight-node isoparametric quadrilateral element (Quad8). The results show that, for regular meshes, the performance of the element is superior to that of Quad4 element, and comparable to that of Quad8 element. For distorted meshes, the present element has better mesh-distortion tolerance than Quad4 and Quad8 elements.

1. Introduction

In the past several decades, the finite element method (FEM) [1] has been extensively used in many fields of engineering [2–4]. Nevertheless, accuracy of some classic isoparametric elements is highly sensitive to mesh distortions [5]. Recently, the meshfree or meshless methods (MMs) which do not need a mesh to discretize the problem domain and therefore are not limited by mesh distortion woes [6], have attracted many researchers. The meshfree methods are very suitable to solve practical problems including large deformation [7] and fracture propagation [8]. Some of the important works associated with meshfree methods are Smoothed Particle Hydrodynamics (SPH) [9], Diffuse Element Method (DEM) [10], Element-Free Galerkin method (EFG) [11], reproducing kernel particle method (RKPM) [12], stable particle methods [13], meshfree local Petrov-Galerkin method (MLPG) [14], point interpolation method (PIM) [15], radial point interpolation method (RPIM) [16] and smoothed point interpolation methods [17]. Like FEM, the meshfree methods either are not free from drawbacks [6]. Shape functions in some of the meshfree methods do not possess the much desired Kronecker delta property which renders the application of boundary condition more difficult than in FEM. The meshfree methods are also computationally more expensive than FEM [6]. As a result, some hybrid schemes [18] have been proposed to improve the properties of meshfree methods.

In recent years, Partition-of-unity (PU) based methods [19] have been developed and successfully used in many fields [20–23]. Notable among these PU based methods are hp-clouds [24], generalized finite element method (GFEM) [25], particle-partition of unity method [26], numerical manifold method [27–31] and extended finite element method (XFEM) [32]. An attractive feature of PU-based methods is that they are capable of constructing a higher order global approximation by simply increasing the order of the local approximation functions without adding new nodes [33]. However, "linear dependence" (LD) problem occurs when both the PU functions and the local functions are taken as explicit polynomials [6,19]. Here, the "linear dependence" (LD) problem means after applying the basic boundary condition to eliminate the rigid body displacement, the global stiffness matrix is still singular. Some effective approaches to eliminate the LD problems can be found in [34,35]. In other front, Liu and his co-workers have developed a family of smoothed finite element methods (S-FEMs), such as cell-based S-FEM (CS-FEM) [36], node-based S-FEM (NS-FEM) [37], edge-based S-FEM (ES-FEM) [38], and face-based S-FEM (FS-FEM) [39] to improve FEM. Thanks to the smoothing technique [40], the S-FEM has "softer" stiffness than FEM, and yields more accurate solutions [36].

In order to synergize the individual strengths of meshfree and

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finite element methods, Rajendran et al. developed a new family of PU-based [19] 'FE-Meshfree' elements [6,33,41] for linear, geometry nonlinear and free vibration analyses. 'FE-Meshfree' elements combine the classical shape functions of isoparametric elements with the shape functions of a meshfree method so as to arrive at hybrid shape functions termed as *composite shape functions* [33]. As a result, these 'FE-Meshfree' elements inherit better accuracy, higher convergence rate, and high tolerance to mesh distortion from the meshfree methods, while preserving the Kronecker-delta property of the standard isoparametric elements. Moreover, these 'FE-Meshfree' elements have been known to be free from the linear dependence problem which otherwise cripples many of the PU-based finite elements [6]. Although 'FE-Meshfree' elements can construct higher order shape functions than classical isoparametric elements, derivatives of *composite shape functions* of 'FE-Meshfree' elements [6,33,41,42] are not continuous at nodes and extra smoothing operations are required to calculate nodal stress in post processing. To further improve the property of 'FE-Meshfree' elements, Tang et al. [43] developed a new hybrid 'FE-Meshfree' four-node quadrilateral element with continuous nodal stress (Quad4-CNS). Furthermore, a hybrid 'FE-Meshfree' three-node triangular element with continuous nodal stress (Trig3-CNS) [44] was developed. These two elements have been successfully used for linear elasticity problems [43,44].

In the present paper, the Quad4-CNS element is extended to free and forced vibration analyses of two dimensional solids. The outline of this paper is as follows. Section 2 briefly reviews the construction of shape functions for the Quad4-CNS element. Section 3 gives the equations for free and forced vibration analyses. Typical numerical tests are carried out to assess accuracy of the proposed Quad4-CNS element in Section 4. Finally, conclusions are drawn in Section 5.

2. Construction shape function for Quad4-CNS

Consider a quadrilateral domain Ω described by four nodes $\{P_1\ P_2\ P_3\ P_4\}$ and introduce an arbitrary point $P(\boldsymbol{x})$ with the coordinates $\boldsymbol{x}=(x,y)$. According to the concept of PUM [19], in the quadrilateral domain Ω, the Quad4-CNS global approximation $u^h(\boldsymbol{x})$ can be represented in the following form:

$$u^h(\boldsymbol{x}) = w_1(\boldsymbol{x})u_1(\boldsymbol{x}) + w_2(\boldsymbol{x})u_2(\boldsymbol{x}) + w_3(\boldsymbol{x})u_3(\boldsymbol{x}) + w_4(\boldsymbol{x})u_4(\boldsymbol{x}) \quad (1)$$

where, $w_i(\boldsymbol{x})$ and $u_i(\boldsymbol{x})$ are the weight functions and the nodal approximations associated with node i.

The weight functions $\{w_i(\boldsymbol{x}), i = 1, 2, 3, 4\}$ with the global Cartesian coordinates are mapped from 'parent' weight functions in the local coordinates [43]. The formulations for coordinate transformation are represented as:

$$x = \tilde{N}_1(\xi,\eta)x_1 + \tilde{N}_2(\xi,\eta)x_2 + \tilde{N}_3(\xi,\eta)x_3 + \tilde{N}_4(\xi,\eta)x_4 \quad (2)$$

$$y = \tilde{N}_1(\xi,\eta)y_1 + \tilde{N}_2(\xi,\eta)y_2 + \tilde{N}_3(\xi,\eta)y_3 + \tilde{N}_4(\xi,\eta)y_4 \quad (3)$$

where $\tilde{N}_1(\xi,\eta)$, $\tilde{N}_2(\xi,\eta)$, $\tilde{N}_3(\xi,\eta)$, $\tilde{N}_4(\xi,\eta)$ are expressed in the following form [1]

$$\tilde{N}_i(\xi,\eta) = (1+\xi_0)(1+\eta_0)/4, \xi_0 = \xi_i\xi, \quad \eta_0 = \eta_i\eta,\ i = 1, 2, 3, 4. \quad (4)$$

Unlike the 'FE-Meshfree' QUAD4 element with least square point interpolation functions (Quad4-LSPIM) [41], which uses the shape functions of Quad4 to define its weight functions, the weight functions of Quad4-CNS element are written as [43]

$$w_i(\xi,\eta) = (1+\xi_0)(1+\eta_0)(2+\xi_0+\eta_0-\xi^2-\eta^2)/8, i = 1, 2, 3, 4. \quad (5)$$

There are three important features for the weight functions of Quad4-CNS element as described in Appendix A.

The nodal approximations associated with node i, as yet unknown, are expressed in the interpolation form as

$$u_i(\boldsymbol{x}) = \sum_{j=1}^{n^{[i]}} \hat{\phi}_j^{[i]}(\boldsymbol{x})a_j, \quad (6)$$

in which $n^{[i]}$ is the total number of nodes in the domain Ω_i, (Fig. B1), a_j is the nodal displacement of node j and $\hat{\phi}_j^{[i]}(\boldsymbol{x})$ is the shape function of the nodal approximation $u_i(\boldsymbol{x})$ associated with node j. (The procedure to obtain u_i is described in Appendix B.)

The Quad4-CNS approximation $u^h(\boldsymbol{x})$ can be represented in a common form:

$$u^h(\boldsymbol{x}) = \sum_{i=1}^{N} \phi_i(\boldsymbol{x})a_i, \quad (7)$$

in which $\phi_i(\boldsymbol{x})$ is the shape function corresponding to the node i. N is the total number of the nodes in domain $\hat{\Omega}$ (Fig. B2). Substitution of Eq. (6) into Eq. (1), and then the Quad4-CNS global approximation can be constructed as

$$u^h(\boldsymbol{x}) = \sum_{i=1}^{4} w_i(\boldsymbol{x}) \sum_{j=1}^{n^{[i]}} \hat{\phi}_j^{[i]}(\boldsymbol{x})a_j. \quad (8)$$

By manipulating Eq. (8), the Quad4-CNS shape functions in Eq. (7) can be represented as

$$\phi_i(\boldsymbol{x}) = w_1(\boldsymbol{x})\hat{\phi}_i^{[1]}(\boldsymbol{x}) + w_2(\boldsymbol{x})\hat{\phi}_i^{[2]}(\boldsymbol{x}) + w_3(\boldsymbol{x})\hat{\phi}_i^{[3]}(\boldsymbol{x}) + w_4(\boldsymbol{x})\hat{\phi}_i^{[4]}(\boldsymbol{x}) \quad (9)$$

If node j is not in the neighboring domain Ω_i, then $\hat{\phi}_j^{[i]}(\boldsymbol{x})$ is defined to be of zero value,

$$\hat{\phi}_j^{[i]}(\boldsymbol{x}) \equiv 0. \quad (10)$$

Some useful properties of Quad4-CNS are shown as follows [43]:

(1) The derivative of weight function is of zero value at the nodes.
(2) The derivative of Quad4-CNS global approximation is continuous at the nodes.
(3) The Kronecker-delta property

$$\phi_i(\boldsymbol{x}_j) = \delta_{ij} \quad (11)$$

3. Forced and free vibration analyses

Consider a 2D problem defined in domain V and let domain V be discretized by a set of non-overlapping quadrilateral domain: $V = \cup_{i=1}^{N} V_i$. Using the Quad4-CNS shape functions derived in Section 2, the discretized equation system of dynamic analysis is obtained as [16,38]

$$\mathbf{M}\ddot{\mathbf{a}} + \mathbf{C}\dot{\mathbf{a}} + \mathbf{K}\mathbf{a} = \mathbf{f} \quad (12)$$

where $\mathbf{K}$, and $\mathbf{M}$ are the global stiffness matrix and global mass matrix, respectively, and defined by

$$\mathbf{K}_{ij} = \sum \mathbf{K}_{ij}^e,\ \mathbf{M}_{ij} = \sum \mathbf{M}_{ij}^e \quad (13)$$

where