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An efficient nodal integration with quadratic exactness for three-dimensional meshfree Galerkin methods

Bingbing Wang [a], Qinglin Duan [a,b,c,*], Yulong Shao [a], Xikui Li [a], Dixiong Yang [a], Hongwu Zhang [a]

[a] The State Key Laboratory of Structural Analysis for Industrial Equipment, Department of Engineering Mechanics, Dalian University of Technology, Dalian 116024, PR China

[b] The State Key Laboratory of Water Resources and Hydropower Engineering Science, Wuhan University, Wuhan 430072, PR China

[c] The State Key Laboratory of Geohazard Prevention and Geoenvironment Protection, Chengdu University of Technology, Chengdu 610059, PR China

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ABSTRACT

Quadratically consistent nodal integration (QCNI) for three-dimensional meshfree Galerkin methods with second order approximation is presented. The number of integration points is dramatically reduced since the weak form is evaluated only at approximation nodes. The stabilization for such reduced integration stems from the correction of nodal derivatives. Such correction is based on the orthogonality condition between the stress and strain difference in the framework of Hu-Washizu three-field variational principle. Taylor series expansion is employed such that a linear strain field in each background integration cell can be exactly reproduced. Three-dimensional quadratic patch test is exactly passed by QCNI and thus it possesses quadratic exactness. In contrast, the stabilized conforming nodal integration (SCNI) which is so far the most successful nodal integration technique can only reproduce a constant strain field in each integration cell and fails to pass the quadratic patch test. The comprehensive superiorities of the proposed QCNI over the existing SCNI in accuracy, convergence, efficiency and smoothness of the resulting stress fields are further demonstrated by several three-dimensional numerical examples. Especially, it is shown in some example that the accuracy of QCNI is surprisingly four order higher than that of SCNI.

1. Introduction

Meshfree Galerkin methods developed in the past twenty years, such as the element-free Galerkin method [1] and the reproducing kernel particle method [2], have several appealing advantages in comparison with the traditional finite element method (FEM) which is the dominant numerical tool for engineering problems. For example, meshfree approximation is constructed entirely in terms of a set of scattered nodes. Elements which stand for pre-defined nodal connectivity are no longer needed. This gives meshfree methods substantial flexibilities in large deformation analysis where element distortion poses a major issue for FEM. In addition, h-adaptivity is easier to be implemented in meshfree methods since only nodes instead of elements need to be added or removed adaptively. Furthermore, meshfree approximation is much smoother than the Lagrangian interpolation of FEM and this may lead to better convergence. It is also much more convenient to construct high order approximation in meshfree methods than that in FEM since the high order elements such as the 20-node hexahedron are completely not needed. As a result, meshfree methods have the potential to perform better than FEM in numerical analysis and thus may challenge the dominance of the latter.

However, so far meshfree methods are still not widely accepted in industry due to their low computational efficiency. Since meshfree approximation functions are non-polynomial rational functions, numerical integration of the weak form requires much more evaluation points in meshfree methods than that in FEM, especially for three-dimensional analysis which is considered in this paper. For example, $5 \times 5 \times 5$ Gauss points per background hexahedral cell are used for the three-dimensional EFG method in Barry and Saigal [3]. It is impossible to use such a large number of integration points in numerical analysis of a real engineering problem. But even worse, these numerous evaluation points still cannot integrate the weak form exactly and therefore the accuracy

* Corresponding author at: The State Key Laboratory of Structural Analysis for Industrial Equipment, Department of Engineering Mechanics, Dalian University of Technology, Dalian 116024, PR China.
E-mail address: qinglinduan@dlut.edu.cn (Q. Duan).

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of the method is not acceptable.

Several strategies were developed in the literature to reduce the number of integration points for meshfree methods, such as the stress-point integration [4,5], the support domain integration [6], etc. Among them, the nodal integration which employs only the approximation nodes as the integration points is the dominant technology and attracts intensive studies. It is noted that two sets of points are used in FEM and meshfree methods with Gauss integration: one set for approximation (interpolation) and the other one for integration, whereas only one set of points is employed in meshfree methods with nodal integration. This makes the method more truly meshless like. Besides, it also brings some merits. For instance, the mapping between approximation nodes and integration points is not needed at all for nodal integration. These are probably the reasons why nodal integration is popular in meshfree fields.

Beissel and Belytschko [7] initialized the study of nodal integration and found that direct nodal integration is unstable and inaccurate. They introduced a least-square stabilization term into the weak form. As a result, stability of the method is remarkably improved. However, its accuracy is still unsatisfactory. Besides, an artificial parameter is involved and its selection depends on numerical experiments. Nagashima [8] introduced the stabilization terms by a Taylor expansion of the stiffness matrix. A merit of this strategy is that it is rational and no artificial parameter is involved. Liu et al. [9] also used this technique to stabilize the nodal integration of the radial point interpolation method (RPIM) and higher order Taylor's expansion is employed. However, instability still may present, especially in stress fields [10].

The most successful and widely applied nodal integration in meshfree fields is the stabilized conforming nodal integration (SCNI) proposed by Chen et al. [11]. According to the satisfaction of the linear patch test condition, they formulated an integration constraint (IC) and further developed a strain smoothing to meet IC exactly. Therefore, linear patch test can be exactly passed by SCNI whereas high order Gauss integration cannot, that is, SCNI dramatically reduces the number of quadrature points and meanwhile effectively improves the accuracy. This leads to a big increase in efficiency. Also, no artificial parameter is involved. Consequently, SCNI has been widely applied in various studies, see [12–18]. The technique of strain smoothing has also been extended to natural-element method [19], radial point interpolation method [20] and even finite element method [21] and has been applied to various 3D problems such as heat transfer [22], adaptive analysis [23,24], fluid-structure interaction (FSI) [25], etc.

However, Puso et al. [26] reported that SCNI may still cause instabilities near domain boundaries. Furthermore, Duan et al. [27] showed that the strain smoothing in SCNI can only reproduce a constant strain field for each background integration cell and therefore SCNI is not adequate for high order meshfree approximation which requires linear and higher order strain fields should be exactly reproduced. To remedy this limitation, they proposed a consistency framework guiding the correction of nodal derivatives based on the divergence theorem between a nodal shape function and its derivatives. The proposed framework can be applied to arbitrary order approximations. For quadratic meshfree approximation, they demonstrated that, under this framework, a three-point integration scheme, namely the QC3 scheme, can be straightforwardly established. This scheme can reproduce a linear strain field for each integration cell. As a consequence, it can pass the quadratic patch test exactly whereas the SCNI cannot. The accuracy, convergence, efficiency and stability are also remarkably improved by QC3 in comparison to those of SCNI. Reformulation of QC3 based on the Hu-Washizu three-field variational principle is presented in [28] and its extension to 3D is given in [29]. However, QC3 is not nodal integration.

To further develop a nodal integration method, Duan et al. [30] introduced the high order derivatives into the consistency framework by using the technique of Taylor series expansion. The basic idea is that a linear strain field can be reproduced for each background integration cell by one first-order and two second-order derivatives at one quadrature point instead of using three points as in QC3. The developed nodal integration method, namely quadratically consistent nodal integration (QCNI), has a similar numerical performance as the QC3 scheme and also possesses quadratic exactness. Thus, it is much better than SCNI which only have linear exactness. Chen et al. [31] also proposed an arbitrary order variationally consistent integration method by correcting the test function in a Petrov-Galerkin weak form. Second order exactness of the method is demonstrated by numerical examples. However, it leads to asymmetric stiffness matrix which is computationally inefficient. Besides, so far the method as well as the QCNI method is still limited to two-dimensional problems.

The purpose of this paper is to extend the QCNI to 3D. In view of the fact that most real industrial problems are in 3D instead of in 2D and meshfree methods with Gauss integration employ much more integration points in 3D than in 2D, such extension is of great significance, especially for the application of the meshfree methods to real industrial problems. To our knowledge, the proposed scheme is the first nodal integration for three-dimensional meshfree Galerkin methods which provides second order exactness.

The remaining paper is structured as follows. The formulation of the three-dimensional EFG method with its MLS approximation and Galerkin discretization is briefly reviewed in Section 2. The existing stabilized conforming nodal integration (SCNI) is outlined in Section 3. Derivation of the proposed QCNI method from the Hu-Washizu three-field variational principle is described in Section 4. The consistency of the corrected nodal derivatives are proved in Section 5. Numerical results are presented in Section 6 followed by the conclusion in Section 7.

2. Element-free Galerkin method: three-dimensional formulations

Consider a three dimensional elastostatic problem in the domain $\Omega \subset \mathbf{R}^3$ discretized by a set of nodes $\mathbf{X}_I$, the approximation of the displacement vector at an arbitrary point $\mathbf{x}$ can be written as

$$\mathbf{u}^h(\mathbf{x}) = \mathbf{N}(\mathbf{x})\mathbf{U} = \sum_I \mathbf{N}_I(\mathbf{x})\mathbf{U}_I \tag{1}$$

where $\mathbf{U}$ is the vector of nodal displacement parameter and $\mathbf{U_I} = \begin{bmatrix} u_I & v_I & w_I \end{bmatrix}^{\mathrm{T}}$. The matrix of moving least square (MLS) nodal shape functions is

$$\mathbf{N}(\mathbf{x}) = \begin{bmatrix} \mathbf{N}_1(\mathbf{x}) & \mathbf{N}_2(\mathbf{x}) & \cdots & \mathbf{N}_n(\mathbf{x}) \end{bmatrix} \tag{2}$$

where n is the number of nodes, $\mathbf{N}_I(\mathbf{x}) = N_I(\mathbf{x})\mathbf{I}_3$ and $\mathbf{I}_3$is the 3×3 identity matrix. The MLS shape function $N_I(\mathbf{x})$ is usually constructed by a weighted interpolation residual as presented in [1]. Here, we follow Liu et al. [32] and Belytschko and Fleming [33] where the shape function is constructed directly from the consistency condition. To begin with this type of presentation, the shape function $N_I(\mathbf{x})$ is written as

$$N_I(\mathbf{x}) = \mathbf{p}^{\mathrm{T}}(\mathbf{X}_I) w_I(\mathbf{x}) \boldsymbol{\alpha}(\mathbf{x}) \tag{3}$$

where $\mathbf{p}(\mathbf{x})$ is a vector of base functions and $w_I(\mathbf{x})$ a weight function. In this paper, the following quadratic base is employed

$$\mathbf{p}(\mathbf{x}) = \begin{bmatrix} 1 & x & y & z & \frac{1}{2}x^2 & \frac{1}{2}y^2 & \frac{1}{2}z^2 & xy & yz & zx \end{bmatrix}^{\mathrm{T}} \tag{4}$$