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Inverse contact problem for an elastic half-space

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ABSTRACT

This paper presents a system of integral equations for the determination of contact stresses on a part of the boundary of elastic half-space by measured data of displacements on the rest of the stress-free boundary. Inverse problems like this are refereed to as conditionally ill-posed with pronounced dependence of the solution from small perturbations in measured data. The 3D problem formulation is based on spatial harmonic functions. It is proposed to use a Trefftz-type method for the sought harmonic functions based on the radial basis functions to solve the system of integral equations. A synthetic example is presented to illustrate the proposed approach.

1. Introduction

Determination of stresses acting in contact zones between two or more bodies is the primary problem of contact mechanics. The classical approach [1] assumes different types of boundary conditions within contact zones. These can be formulated in terms of stresses for soft punches, displacement for rigid punches or certain linear relationships between stresses and displacements to take friction into account. In all these cases the correspondent boundary value problem is well posed, thus, it possesses a unique and stable solution.

An alternative approach to contact problems deals with so-called, ill-posed problems, in which no boundary conditions are specified in a contact zone but some excessive data are given (or measured) on the rest of the boundary. In particular, if all components of displacements are measured on a stress-free boundary, then six boundary conditions (three homogeneous for the stress vector and three non-homogeneous for the displacement vector) may be imposed outside the contact zone, which make the boundary problem to be overspecified outside the contact zone and underspecified inside it. Perhaps, the first comprehensive analysis of solvability of such problems for elastic domains has been conducted by Shvab [2]. On the other hand, this problem can be reformulated in terms of holomorphic vectors, on which the proof of uniqueness for the Cauchy problem has been reported in another paper of the same author [3] (note different spelling of the surname as a result of translation). The author has also admitted fair results of the solution obtained without regularisation.

Regularisation techniques have been developed to address different engineering and geophysics inverse (incorrectly posed) problems to obtain stable solutions. Ill-posed problems are generally defined as the problems with no unique or unstable solutions [4]; they require the application of special methods of solutions; for instance, the methods based on Tikhonov's regularisation have been reported to be an effective tool for obtaining stable solutions in many fields of applied mathematics [4,5]. In solid mechanics ill-posed formulations are less developed, although there is a wide class of applications (as identification of elastic properties, defects or buried objects, e.g. cracks or inclusions) that are incorrectly posed mathematically, see review by Bonnet and Constantinescu [6]. The mentioned problems can be effectively reduced to the formulation with boundary conditions overspecified (the number of scalar conditions is greater than the problem dimension; e.g., for spatial elasticity it is greater than three) on a part of the boundary and underspecified on its rest. The methods applied to handle such formulations are diverse, but many of them employ the methods based on integral equations, e. g, Tanaka and Masuda [7] for identification of flaw shape in a body, Kubo [8] for overview of inverse problems in fracture mechanics, Zabaras [9] for contact problems where measurements at the contact area are difficult, Gao and Mura [10] for taking into account plastic flow near crack tips, Hsieh and Mura [11] for nondestructive cavity identification, Galybin [12] for analytic solutions in half-plane for surface subsidence monitoring, Yeih et al. [13] and Bui [14] for providing Fredholm integral equations, Simpson et al. [15] for the development of special elements for contact problems. Finite element formulations are also frequently used, e.g. [16–18]. Methods of complex variables for plane problems have been applied in, e.g., Galybin [19] and Tsvelodub [20].

Analytical solutions are rarely available for ill-posed formulations (with exceptions for simple domains, e.g. for wedge-like domains [19]), therefore the development of stable numerical

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methods has recently become the main focus of different research. One should acknowledge contribution made by the researches from University of Leeds in the development of regularisation techniques, iterative methods and algorithms for solving non-classical boundary value problems [21–23]. In particular it has been shown that the use of the SVD regularisation presents a valuable computational tool in elastostatics, e.g. [24,25]. It will used further to present numerical results for the problem formulated below.

2. Problem formulation

Let an isotropic elastic half-space, $x_3 \le 0$, be loaded on a part of the boundary by a punch (elastic, rigid or soft). The magnitudes of the contact stresses and the shape of the punch (both its profile and the contact zone) are, in general, unknown. Instead, the displacements are known on the free part of the boundary $x_3=0$ outside the contact zone. It is further assumed that the action of the punch is replaced by certain unknown stress distribution over a planar subdomain Ω bellowing to the boundary $x_3=0$ that is referred to as a contact zone. The elastic constants are denoted further as G for the modulus of rigidity and ν for Poisson's ratio. We will further normalise the stress components by doubled shear modulus, i.e. suppose $2G=1$ for compactness.

The problem consists of finding the contact stress distribution in the contact zone Ω, which in turn makes possible the determination of the stress and the displacements fields in the entire domain.

Six symmetric components of the stress tensor, σ_{ij} ($i,j=1,2,3$), satisfy three differential equations of equilibrium (DEE) written down without taking into account the body forces

$$\partial_j \sigma_{ij} = 0, \quad i, j = 1, 2, 3 \tag{1}$$

They are connected to the components of displacements, u_i ($i=1,2,3$), by the Hooke's law for small strains

$$\frac{\sigma_{ij}}{2G} = \frac{\nu}{1-2\nu}\delta_{ij} e + \frac{1}{2}(\partial_i u_j + \partial_j u_i), \qquad e = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3, \quad i, j = 1, 2, 3 \tag{2}$$

Hereafter the symbols ∂_i, ∂_{ij} and ∂_{ijk} stand for partial derivatives or the first and higher orders respectively with respect to the variables shown in subscript ($i,j,k=1,2,3$); δ_{ij} is the Kronecker-delta ($\delta_{ij}=1$, $\delta_{ij}=0$, $i\neq j$).

The boundary conditions assume the form

$$\begin{aligned} &u_1(x_1, x_2, 0) = u_1^0(x_1, x_2), \quad u_2(x_1, x_2, 0) = u_2^0(x_1, x_2), \quad u_3(x_1, x_2, 0) = u_3^0(x_1, x_2); \\ &\sigma_{33}(x_1, x_2, 0) = \sigma_{13}(x_1, x_2, 0) = \sigma_{23}(x_1, x_2, 0) = 0 \quad \text{on} \quad x_3 = 0, \quad (x_1, x_2) \notin \Omega \end{aligned} \tag{3}$$

where the functions u_j^0 are known on the stress-free boundary (measured, in general, with certain errors). It should be noted that the partial derivatives of the boundary values with respect to the variables (x_1, x_2) are also known, which will be used further on.

We also assume that the following integrals representing the resulting force $\mathbf{P}=(P_1,P_2,P_3)$ may be known

$$\iint_\Omega \sigma_{13}(x_1, x_2, 0)\, dx_1\, dx_2 = P_1, \quad \iint_\Omega \sigma_{23}(x_1, x_2, 0)\, dx_1\, dx_2 = P_2, \quad \iint_\Omega \sigma_{33}(x_1, x_2, 0)\, dx_1\, dx_2 = P_3 \tag{4}$$

The general solution for a 3D domain can be expressed in terms of 3 independent harmonic functions. Based on the Trefftz integral [26] it is possible to derive the following relationships for the components of displacements and stresses in terms of harmonic functions (f,g,h), see detail in Sih and Liebowitz [27]

$$u_1 = (1-2\nu)\partial_1 f - 2(1-\nu)\partial_3 g + x_3\partial_1(\partial_3 f + \partial_1 g + \partial_2 h) \tag{5}$$

$$u_2 = (1-2\nu)\partial_2 f - 2(1-\nu)\partial_3 h + x_3\partial_2(\partial_3 f + \partial_1 g + \partial_2 h) \tag{6}$$

$$u_3 = -2(1-\nu)\partial_3 f - (1-2\nu)(\partial_1 g + \partial_2 h) + x_3\partial_3(\partial_3 f + \partial_1 g + \partial_2 h) \tag{7}$$

$$\sigma_{11} = \partial_{11} f + 2\nu\partial_{22} f - 2(1-\nu)\partial_{13} g - 2\nu\partial_3(\partial_1 g + \partial_2 h) + x_3\partial_{11}(\partial_3 f + \partial_1 g + \partial_2 h) \tag{8}$$

$$\sigma_{22} = \partial_{22} f + 2\nu\partial_{11} f - 2(1-\nu)\partial_{23} g - 2\nu\partial_3(\partial_1 g + \partial_2 h) + x_3\partial_{22}(\partial_3 f + \partial_1 g + \partial_2 h) \tag{9}$$

$$\sigma_{33} = -\partial_{33} f + x_3\partial_{33}(\partial_3 f + \partial_1 g + \partial_2 h) \tag{10}$$

$$\sigma_{12} = (1-2\nu)\partial_{12} f - (1-\nu)\partial_3(\partial_1 g + \partial_2 h) + x_3\partial_{12}(\partial_3 f + \partial_1 g + \partial_2 h) \tag{11}$$

$$\sigma_{13} = -(1-\nu)\partial_{33} g + \nu\partial_1(\partial_1 g + \partial_2 h) + x_3\partial_{13}(\partial_3 f + \partial_1 g + \partial_2 h) \tag{12}$$

$$\sigma_{23} = -(1-\nu)\partial_{33} h + \nu\partial_2(\partial_1 g + \partial_2 h) + x_3\partial_{23}(\partial_3 f + \partial_1 g + \partial_2 h) \tag{13}$$

It should be noted that the function f addresses the case of normal stresses applied to $x_3=0$ with zero shear stresses, while the functions g,h present the general solution for the case when the shear stresses are applied on the boundary and the normal stresses are absent.

The following equality is valid for any harmonic function H everywhere in the domain including its boundary:

$$\partial_{11} H + \partial_{22} H = -\partial_{33} H \tag{14}$$

It will be used in subsequent derivations where necessary.

The derivatives of the shear stress components are found as follows:

$$\partial_1\sigma_{13} = -(1-\nu)\partial_{133} g + \nu\partial_{11}(\partial_1 g + \partial_2 h) + x_3\partial_{113}(\partial_3 f + \partial_1 g + \partial_2 h) \tag{15}$$

$$\partial_2\sigma_{13} = -(1-\nu)\partial_{233} g + \nu\partial_{12}(\partial_1 g + \partial_2 h) + x_3\partial_{123}(\partial_3 f + \partial_1 g + \partial_2 h) \tag{16}$$

$$\partial_1\sigma_{23} = -(1-\nu)\partial_{133} h + \nu\partial_{12}(\partial_1 g + \partial_2 h) + x_3\partial_{123}(\partial_3 f + \partial_1 g + \partial_2 h) \tag{17}$$

$$\partial_2\sigma_{23} = -(1-\nu)\partial_{233} h + \nu\partial_{22}(\partial_1 g + \partial_2 h) + x_3\partial_{223}(\partial_3 f + \partial_1 g + \partial_2 h) \tag{18}$$

Using (15)–(18) one can form the following combinations:

$$\partial_1\sigma_{23} - \partial_2\sigma_{13} = -(1-\nu)\partial_{33}(\partial_1 g - \partial_2 h) \tag{19}$$

$$\partial_1\sigma_{13} + \partial_2\sigma_{23} = -(1-\nu)\partial_{33}(\partial_1 g + \partial_2 h) - \nu\partial_{33}(\partial_1 g + \partial_2 h) - x_3\partial_{333}(\partial_3 f + \partial_1 g + \partial_2 h) \tag{20}$$

The derivatives of the first two components of the displacements are as follows:

$$\partial_1 u_1 = (1-2\nu)\partial_{11} f - 2(1-\nu)\partial_{31} g + x_3\partial_{11}(\partial_3 f + \partial_1 g + \partial_2 h) \tag{21}$$

$$\partial_2 u_1 = (1-2\nu)\partial_{12} f - 2(1-\nu)\partial_{23} g + x_3\partial_{12}(\partial_3 f + \partial_1 g + \partial_2 h) \tag{22}$$

$$\partial_1 u_2 = (1-2\nu)\partial_{12} f - 2(1-\nu)\partial_{13} h + x_3\partial_{12}(\partial_3 f + \partial_1 g + \partial_2 h) \tag{23}$$

$$\partial_2 u_2 = (1-2\nu)\partial_{22} f - 2(1-\nu)\partial_{23} h + x_3\partial_{22}(\partial_3 f + \partial_1 g + \partial_2 h) \tag{24}$$

It follows from (21)–(24) that

$$\partial_1 u_2 - \partial_2 u_1 = 2(1-\nu)\partial_3(\partial_2 g - \partial_1 h) \tag{25}$$