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Singular boundary method using time-dependent fundamental solution for transient diffusion problems

Wen Chen*, Fajie Wang

State Key Laboratory of Hydrology-Water Resources and Hydraulic Engineering, Center for Numerical Simulation Software in Engineering and Sciences, College of Mechanics and Materials, Hohai University, Nanjing 210098, China

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ABSTRACT

This paper documents the first attempt to apply the singular boundary method (SBM) with time-dependent fundamental solution to transient diffusion equations. An inverse interpolation technique is introduced to determine the origin intensity factor of the SBM. The scheme is mathematically simple, easy-to-program, truly boundary-only, free of integration and mesh. Several examples, especially three-dimensional (3D) cases, are provided to verify time-dependent SBM strategy. The numerical results clearly demonstrate its great potential.

1. Introduction

Diffusion equations model a large number of physical phenomena in many different areas of mathematical physics, applied science, and engineering. For example, image processing [1–3], microwave heating process [4,5], spontaneous ignition [6], and mass transport in groundwater [7]. The numerical investigation of diffusion problems is of great interest for all disciplines in engineering and science. A large variety of methods is available for the solution of such problems and each of them offers certain advantages and disadvantages. The boundary element method (BEM) [8–10] has become increasingly attractive to scientists and engineers since it has certain advantages over other methods such as the finite element method (FEM) and finite difference method (FDM) [11,12]. One of these advantages is obviously its ability in reducing the dimensionality of a problem by one. However, the attractiveness of the BEM is generally lost when it is used to solve diffusion problems with source terms, since singular domain integrals usually appear in such a BEM formulation. One of the solutions, among many others, is to adopt the so-called dual reciprocity method (DRM) [13] combined with the finite difference scheme to deal with the time derivative of the governing equation [14–19]. Nevertheless, as far as numerical efficiency is concerned, the finite difference schemes can be quite time-consuming, especially when nonlinear iterations must be performed at each time step for a nonlinear problem. However, the main drawback of the BEM is the determination of the singular integrals at the boundaries, which requires a great amount of computational effort especially for the 3D problems. The method of fundamental solutions (MFS) [20–22] has emerged as a boundary only collocation method with the merit of easy programming, high accuracy, and fast convergence. During the past decade, this method has been successfully applied to the solution of various partial differential equations. However, a fictitious boundary slightly outside the problem domain is required in order to place the source points and avoid the singularity of the fundamental solution. The determination of the distance between the real boundary and the fictitious boundary is based on experience and therefore troublesome. In recent years, various efforts have been made aiming to remove this barrier in the MFS, so that the source points can be placed on the real boundary directly [23–26].

The singular boundary method (SBM) [26–28] is a recent meshless boundary collocation method for the numerical solution of partial differential equations. The advantages of the SBM over the more classical domain or boundary discretization methods can be summarized as follows: (1) it is a boundary-type method which means that it shares the same advantages of the BEM over domain discretization methods. (2) It is meshless and does not require the task of domain and/or boundary meshing which can be arduous, time-consuming and computationally expensive for problems in complex geometries and high dimensions. (3) It does not involve costly integrations which could be otherwise troublesome as, for example, in

* Corresponding author.
E-mail address: chenwen@hhu.edu.cn (W. Chen).

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the case of the BEM-based methods. (4) The method sidesteps the perplexing fictitious boundary issue associated with the traditional MFS [20–22,29], while retaining the merits of the latter of being truly meshless, mathematically simple and easy-to-program. All these advantages of the SBM have made the method a competitive alternative for certain boundary value problems. This method has been successfully tried for the potential [30], elasticity [31], Helmholtz [28,32], Stokes flow [33], etc. However, these schemes only employ the space-dependent fundamental solutions of the partial differential equations. So far, this method has not been used to analyze the transient problems with time-dependent fundamental solutions.

The purpose of this study is to extend the SBM for the solution of transient diffusion problems with the time-dependent fundamental solution. Consequently, the present SBM scheme does not need additional technique such as the Laplace transform or the finite difference method to handle the time-derivative term. A brief outline of the rest of this paper is as follows. Section 2 introduces the governing equations and the boundary/initial conditions of diffusion problems. The SBM formulation and its numerical implementation with time-dependent fundamental solution are presented in Section 3. The discussion on the comparison of the present results with the analytical results for the test cases is given in Section 4. Finally, some conclusions and remarks are provided in Section 5.

2. Mathematical formulation

Consider a linear diffusion equation that has to be solved over a bounded domain Ω, and assume that Ω is bounded by a boundary $\partial\Omega = \Gamma$. The governing equation is given as

$$\frac{\partial u(\boldsymbol{x}, t)}{\partial t} = k\nabla^2 u(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \Omega, \tag{1}$$

where $\boldsymbol{x}$ is the general spatial coordinate, t is the time, k is the diffusion coefficient and u is the scalar variable to be determined. In this study, a simple boundary condition such as the Dirichlet boundary condition is considered as a first attempt, since a new numerical method based on SBM is being tested for diffusion problems with time-dependent fundamental solution. The initial condition and Dirichlet boundary condition can be expressed as:

$$u(\boldsymbol{x}, t_0) = \bar{u}_0(\boldsymbol{x}, t_0), \quad \boldsymbol{x} \in \Omega \cup \Gamma, \tag{2}$$

$$u(\boldsymbol{x}, t) = \bar{u}(\boldsymbol{x}, t), \quad t \ge t_0, \quad \boldsymbol{x} \in \Gamma, \tag{3}$$

where t_0 is initial time, $\boldsymbol{n}$ denotes the outward normal vector, the overline quantities $\bar{u}_0$ and $\bar{u}$ indicate the given values.

The fundamental solution of governing Eq. (1) for linear diffusion problems is given by

$$u^*(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) = \frac{e^{\frac{-|\boldsymbol{x}-\xi|^2}{4k(t-\tau)}}}{[4\pi k(t-\tau)]^{n/2}} H(t-\tau), \tag{4}$$

where $|\boldsymbol{x} - \xi|$ denotes the Euclidean distance between the point $\boldsymbol{x}$ and the point ξ in R^n, n is the spatial dimension number and $H(t)$ is the Heaviside step function.

3. The singular boundary method and its numerical implementation

3.1. The SBM for diffusion equations

Similar to the MFS, the SBM uses the time-dependent fundamental solution of diffusion equation as the basis function of its

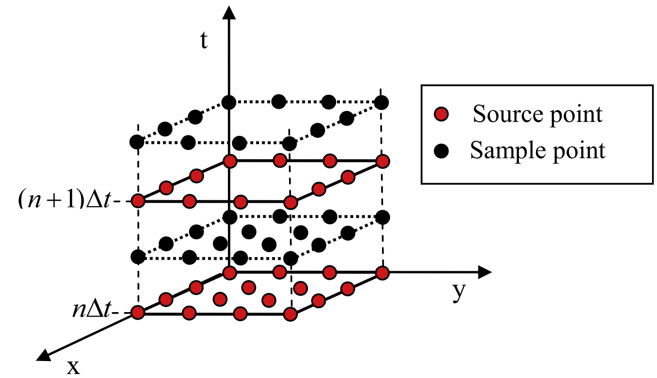

Fig. 1. Distributions of the source points and sample points on the space-time domain for 2D problems.

interpolation. In contrast to the MFS, the SBM sidesteps the perplexing fictitious boundary issue associated with the former by means of the introduction of origin intensity factors, a numerical strategy that isolates the singularities of the fundamental solutions and allows the coincidence of the source and collocation points (see Fig. 1). It is worth noting that the collocation points are placed on the whole domain ($\Omega \cup \Gamma$) at the initial time $t = t_0$ and on the boundary portion (Γ) at $t = t_0 + \Delta t$. The numerical results at the interior domain Ω at $t = t_0 + \Delta t$ obtained by using the SBM will be considered as the initial condition in the next step. This procedure can be repeated until the terminal time is approached. With this idea in mind, the SBM interpolation for diffusion equations can be expressed as

$$u(\boldsymbol{x}_i, t_i) = \sum_{j=1, j\neq i}^{N=N_1+N_2} \alpha_j u^*(\boldsymbol{x}_i, t_i; \boldsymbol{\xi}_j, \tau_j) + \alpha_i A_{ii}\ , \tag{5}$$

where $\boldsymbol{x}_i$ represents the location of the field points and $\boldsymbol{\xi}_j$ gives the location of the source points. t_i and τ_j are the time of the field and source points, respectively, N_1 and N_2 are the number of initial and boundary source points and α_j are the undetermined coefficients. A_{ii} are defined as the origin intensity factors, i.e., the diagonal elements of the SBM interpolation matrix. Observe that when points $\boldsymbol{x}_i$ and $\boldsymbol{\xi}_j$ coincide, meanwhile t_i and τ_j coincide ($i = j$), the term A_{ii} have a singularity of order $(1/(t-\tau)^{n/2})$. The key point in achieving the required accuracy and efficiency of the SBM is the accurate evaluation of such singular terms, i.e., the origin intensity factors mentioned above.

3.2. Inverse interpolation technique

The origin intensity factors A_{ii} for Dirichlet boundary conditions present a weak singularity and can be calculated accurately using the so-called inverse interpolation technique (IIT) proposed in Ref. [26], which can be summarized by the following steps.

Step 1. The IIT requires choosing a known sample solution $\bar{u}_s(\boldsymbol{x}, t)$, such as

$$\bar{u}_s(x_1, x_2, t) = (\sin(x_1) + \sin(x_2))e^{-kt} + 1, \quad for\ 2D, \tag{6}$$

$$\bar{u}_s(x_1, x_2, x_3, t) = (\sin(x_1) + \sin(x_2) + \sin(x_3))e^{-kt} + 1, \quad for\ 3D, \tag{7}$$

then some sample points $\boldsymbol{y}_k$ need to be located inside the physical domain. It is noted that the sample points $\boldsymbol{y}_k$ do not coincide with the source points ξ_j, and the sample points number N_k should not be fewer than the physical boundary source node number N. Fig. 1 shows the location of the sample points, it can be seen the sample