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The plane waves method for axisymmetric Helmholtz problems

Andreas Karageorghis

Department of Mathematics and Statistics, University of Cyprus/ΠΑΝΕΠΙΣΤΗΜΙΟ ΚΥΠΡΟΥ, P.O. Box 20537, Nicosia/ΛΕΥΚΩΣΙΑ, Cyprus

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ABSTRACT

The plane waves method is employed for the solution of Dirichlet and Neumann boundary value problems for the homogeneous Helmholtz equation in two- and three-dimensional domains possessing radial symmetry. The appropriate selection of collocation points and unitary direction vectors in the method leads to circulant and block circulant coefficient matrices in two and three dimensions, respectively. We propose efficient matrix decomposition algorithms which make use of fast Fourier transforms for the solution of the systems resulting from such a discretization. In conjunction with the method of particular solutions, the method is extended to the solution of inhomogeneous axisymmetric Helmholtz problems. Several numerical examples are presented.

1. Introduction

The plane waves method (PWM) is a meshless Trefftz method applicable to the solution of the Helmholtz equation introduced in [1,2,20], see also [7]. We shall study the method with respect to the boundary value problems

$$\Delta u + \kappa^2 u = 0 \quad \text{in } \Omega, \tag{1.1a}$$

subject to the Dirichlet boundary condition

$$u = g_1 \quad \text{on } \partial\Omega, \tag{1.1b}$$

or the Neumann boundary condition

$$\frac{\partial u}{\partial \mathrm{n}} = g_2 \quad \text{on } \partial\Omega, \tag{1.1c}$$

in an axisymmetric domain Ω in $\mathbb{R}^s$, $s = 2, 3,$ where $\mathbf{n}(x, y) = (\mathrm{n}_x, \mathrm{n}_y)$ for $s=2$ and $\mathbf{n}(x, y, z) = (\mathrm{n}_x, \mathrm{n}_y, \mathrm{n}_z)$ for $s=3$ denotes the outward unit normal vector to the boundary at the point (x,y) or (x, y, z). Also, g_1 and g_2 are known functions.

We shall propose efficient matrix decomposition algorithms (MDAs) [4,5] for both the two- and the three-dimensional PWD discretizations of problems (1.1). These MDAs are similar to the ones proposed in the context of the method of fundamental solutions (MFS) for axisymmetric problems [14,15], see also [10]. The proposed MDAs rely on the fact that, for appropriate choices of collocation points and unitary direction vectors in the PWM, the coefficient matrices for the Dirichlet and Neumann boundary value problems (1.1) in two and three dimensions are circulant [6] and block circulant, respectively. The observation that the coefficient matrix in the two-dimensional Dirichlet problem in a disk is circulant was first made in [12] when the method was applied to the modified Helmholtz equation and in [3] in relation to the calculation of the eigenfrequencies of the Laplace operator. It should be noted that the PWM has also been used for the numerical solution of inverse problems in [8]. Moreover, the MFS was applied to Helmholtz problems in [18,19] while the method of particular solutions, which we use for inhomogeneous problems, was applied to Helmholtz problems in [16,17].

The paper is organized as follows. In Section 2 we describe the PWM for the efficient solution of two- and three-dimensional Helmholtz problems possessing radial symmetry. Several numerical examples are presented in Section 3. In Section 4 we extend the proposed algorithms for the efficient solution of inhomogeneous two- and three-dimensional Helmholtz problems possessing radial symmetry. The results of numerical experiments for inhomogeneous problems are presented in Section 5 and, finally, some concluding remarks are given in Section 6.

2. The plane waves method

In the PWM [2], we approximate the solution u of boundary

E-mail address: andreask@ucy.ac.cy

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value problem (1.1) by a linear combination of plane waves

$$u_{\mathcal{J}}(\boldsymbol{x}) = \sum_{j=1}^{\mathcal{J}} c_j \mathrm{e}^{\mathrm{i}\kappa \boldsymbol{x}\cdot\boldsymbol{d}_j}, \quad (x, y) \in \overline{\Omega}, \ \mathrm{i}^2 = -1. \tag{2.1}$$

In (2.1), the $\boldsymbol{d}_j \in \mathbb{R}^s$, $s = 2, 3$, are unitary direction vectors and, clearly, each plane wave in the above expansion satisfies the Helmholtz equation (1.1a), see e.g. [11]. As a result, in order to determine the unknown coefficients $\{a_j\}_{j=1}^{\mathcal{J}}$ we only need to satisfy the boundary conditions of the boundary value problem in question. Density results regarding approximation (2.1) may be found in [2] where it is also shown that the PWM may be viewed as an asymptotic version of the MFS.

2.1. Circular domains

We first consider the simplest case $s=2$, in which Ω is a disk of, say, radius R. We place $\mathcal{J} = N$ collocation points $\{(x_n, y_n)\}_{n=1}^{N}$ on $\partial\Omega$

$$x_n = R\ \cos\ \vartheta_n, \quad y_n = R\ \sin\ \vartheta_n, \quad \text{where } \vartheta_n = 2\pi(j-1)/N, \quad n = 1, \ldots, N. \tag{2.2}$$

We choose the vectors $\{\boldsymbol{d}_n\}_{n=1}^{N} = \left\{\left(d_{x_n}, d_{y_n}\right)\right\}_{n=1}^{N}$ to be the normal unit vectors at the collocation points, i.e.

$$d_{x_n} = \mathrm{n}_{x_n} = \cos\ \vartheta_n, \quad d_{y_n} = \mathrm{n}_{y_n} = \sin\ \vartheta_n, \quad \text{where } n = 1, \ldots, N. \tag{2.3}$$

Using approximation (2.1) with $\mathcal{J}$ replaced by N, the satisfaction of the boundary condition (1.1b) at the collocation points yields

$$\sum_{n=1}^{N} c_n \mathrm{e}^{\mathrm{i}\kappa \boldsymbol{x}_m\cdot\boldsymbol{d}_n} = g_1(\boldsymbol{x}_m), \quad m = 1, \ldots, N, \tag{2.4}$$

or an $N \times N$ system of the form

$$A\ \boldsymbol{c} = \boldsymbol{g}_1, \tag{2.5}$$

where $A_{mn} = \mathrm{e}^{\mathrm{i}\kappa \boldsymbol{x}_m\cdot\boldsymbol{d}_n} = \mathrm{e}^{\mathrm{i}\kappa R\cos(\vartheta_m - \vartheta_n)}$, since $\boldsymbol{x}_m\cdot\boldsymbol{d}_n = R\ \cos\ \vartheta_m\ \cos\ \vartheta_n + R\ \sin\ \vartheta_m\ \sin\ \vartheta_n = R\cos(\vartheta_m - \vartheta_n)$. As A_{mn} depends only on $m - n$, the matrix A is circulant [6]. In the case of a Neumann boundary value problem, collocation of (1.1c) yields

$$\sum_{n=1}^{N} c_n \frac{\partial \mathrm{e}^{\mathrm{i}\kappa \boldsymbol{x}_m\cdot\boldsymbol{d}_n}}{\partial \mathrm{n}} = g_2(\boldsymbol{x}_m), \quad m = 1, \ldots, N \tag{2.6}$$

or a system of the form

$$A\ \boldsymbol{c} = \boldsymbol{g}_2, \tag{2.7}$$

where

$$A_{mn} = \frac{\partial \mathrm{e}^{\mathrm{i}\kappa \boldsymbol{x}_m\cdot\boldsymbol{d}_n}}{\partial \mathrm{n}} = \mathrm{i}\,\kappa\ \mathrm{e}^{\mathrm{i}\kappa \boldsymbol{x}_m\cdot\boldsymbol{d}_n}(d_{x_n}\mathrm{n}_{x_m} + d_{y_n}\mathrm{n}_{y_m}) = \mathrm{i}\,\kappa\ \mathrm{e}^{\mathrm{i}\kappa R\cos(\vartheta_m - \vartheta_n)}\cos(\vartheta_m - \vartheta_n)$$

since $(d_{x_n}\mathrm{n}_{x_m} + d_{y_n}\mathrm{n}_{y_m}) = \cos(\vartheta_m - \vartheta_n)$. As, again, A_{mn} depends only on $m - n$, the matrix A is circulant.

In both cases, the MDA is very simple and well-documented [14]. If we define the unitary $N \times N$ Fourier matrix

$$U_N = \frac{1}{\sqrt{N}}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \bar{\omega} & \bar{\omega}^2 & \cdots & \bar{\omega}^{N-1} \\ 1 & \bar{\omega}^2 & \bar{\omega}^4 & \cdots & \bar{\omega}^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \bar{\omega}^{N-1} & \bar{\omega}^{2(N-1)} & \cdots & \bar{\omega}^{(N-1)(N-1)} \end{pmatrix}, \quad \text{where } \omega = \mathrm{e}^{2\pi\mathrm{i}/N}, \tag{2.8}$$

and let $A = \mathrm{circ}(a_1, a_2, \ldots, a_N)$, then

$$U_N A U_N^* U_N\ \boldsymbol{c} = D\ \tilde{\boldsymbol{c}} = U_N\ \boldsymbol{g} = \tilde{\boldsymbol{g}}, \tag{2.9}$$

where $\tilde{\boldsymbol{c}} = U_N\boldsymbol{c}$, $\tilde{\boldsymbol{g}} = U_N\boldsymbol{g}$ and $\boldsymbol{g}$ is either $\boldsymbol{g}_1$ or $\boldsymbol{g}_2$. Moreover, the matrix $D = U_N A U_N^*$ is diagonal with elements $D = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, where [6]

$$\lambda_n = \sum_{k=1}^{N} a_k \omega^{(k-1)(n-1)}, \quad n = 1, 2, \ldots, N. \tag{2.10}$$

From system (2.9) we have that $\tilde{c}_n = \tilde{g}_n/\lambda_n$, $n = 1, \ldots, N$. Having obtained $\tilde{\boldsymbol{c}}$ we recover $\boldsymbol{c}$ from $\boldsymbol{c} = U_j^* \tilde{\boldsymbol{c}}$.

In conclusion, the MDA can be summarized as follows:

Algorithm 1.

Step 1: `Compute` $\tilde{\boldsymbol{g}} = U_N\,\boldsymbol{g}$.
Step 2: `Construct the diagonal matrix` D `from` (2.10).
Step 3: `Compute` $\tilde{\boldsymbol{c}}$ `from` $\tilde{c}_n = \tilde{g}_n/\lambda_n$, $n = 1, \ldots, N$.
Step 4: `Recover the vector` $\boldsymbol{c}$ `from` $\boldsymbol{c} = U_N^*\tilde{\boldsymbol{c}}$.

In Steps 1, 2 and 4 fast Fourier transforms (FFTs) are used via the MATLAB© [13] commands `fft` and `ifft`. Therefore, the total cost of the algorithm is $O(N\ \log\ N)$.

2.2. Axisymmetric domains

We next consider the three-dimensional axisymmetric case $s=3$, in which Ω is an axisymmetric solid with boundary surface $\partial\Omega$ generated by the rotation of the plane curve Υ about the z-axis. We first place N collocation points on Υ with coordinates $(\mathrm{r}_n, \mathrm{z}_n)$, $n = 1, \ldots, N$, where r_n is the perpendicular distance from the z-axis (i.e. the x-coordinate in $\mathbb{R}^2$) and z_n is the z-coordinate. The $\mathcal{J} = MN$ collocation points on the surface $\partial\Omega$ are then defined from

$$x_{mn} = \mathrm{r}_n\ \cos\ \varphi_m, \quad y_{mn} = \mathrm{r}_n\ \sin\ \varphi_m, \quad z_{mn} = \mathrm{z}_n, \quad \text{where } \varphi_m = 2\pi(m-1)/M, \ n = 1, \ldots, N. \tag{2.11}$$

Note that no collocation points are taken on the z-axis as, due to their invariance under rotation, this would lead to a singular matrix.

We choose the vectors $\{\boldsymbol{d}_{mn}\}_{m,n=1}^{M,N} = \left\{\left(d_{x_{mn}}, d_{y_{mn}}, d_{z_{mn}}\right)\right\}_{m,n=1}^{M,N}$ to be the normal unit vectors at the collocation points, i.e.

$$d_{x_{mn}} = \mathrm{n}_{x_{mn}}, \quad d_{y_{mn}} = \mathrm{n}_{y_{mn}}, \quad d_{z_{mn}} = \mathrm{n}_{z_{mn}}, \quad \text{where } m = 1, \ldots, M, \ n = 1, \ldots, N. \tag{2.12}$$

Since the surface $\partial\Omega$ is axisymmetric, if the normal derivative in the plane at the point $(\mathrm{r}_n, \mathrm{z}_n)$ on Υ is $\boldsymbol{\nu} = (\nu_{\mathrm{r}_n}, \nu_{\mathrm{z}_n})$, then

$$d_{x_{mn}} = \mathrm{n}_{x_{mn}} = \nu_{\mathrm{r}_n}\ \cos\ \varphi_m, \quad d_{y_{mn}} = \mathrm{n}_{y_{mn}} = \nu_{\mathrm{r}_n}\ \sin\ \varphi_m, \quad d_{z_{mn}} = \mathrm{n}_{z_{mn}} = \nu_{\mathrm{z}_n}, \tag{2.13}$$

where $m = 1, \ldots, M$, $n = 1, \ldots, N$.

Instead of (2.1) we write

$$u_{MN}(\boldsymbol{x}) = \sum_{m=1}^{M}\sum_{n=1}^{N} c_{mn}\mathrm{e}^{\mathrm{i}\kappa \boldsymbol{x}\cdot\boldsymbol{d}_{mn}}, \quad (x, y, z) \in \overline{\Omega}. \tag{2.14}$$

In the Dirichlet problem, collocation of boundary condition (1.1b) yields

$$\sum_{m_2=1}^{M}\sum_{n_2=1}^{N} c_{mn}\mathrm{e}^{\mathrm{i}\kappa \boldsymbol{x}_{m_1n_1}\cdot\boldsymbol{d}_{m_2n_2}} = g_1(\boldsymbol{x}_{m_1n_1}), \quad m_1 = 1, \ldots, M, \quad n_1 = 1, \ldots, N, \tag{2.15}$$