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A new approach to non-homogeneous hyperbolic boundary value problems using hybrid-Trefftz stress finite elements

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ABSTRACT

A new approach to the solution of non-homogeneous hyperbolic boundary value problems is casted here using the hybrid-Trefftz stress/flux elements. Similarly to the Dual Reciprocity Method, the technique adopted in this paper uses a Trefftz-compliant set of functions to approximate the complementary solution of the problem and an additional trial basis to approximate its particular solution. However, the particular and complementary solutions are obtained here in a single step, and not sequentially, as typical of the Dual Reciprocity Method. The trial functions used for both particular and complementary solutions are merged together in the same basis and offered full flexibility to combine so as to recover the enforced equations in the best possible way. This option enables Trefftz-compliant functions to compensate for deficiencies of the particular solution basis, meaning that accurate total solutions can be obtained with relatively poor particular solution approximations. The formulation preserves the Hermitian, sparse and localized structure that typifies the matrix of coefficients of hybrid-Trefftz finite elements and avoids the drawbacks of the collocation procedures that arise in the Dual Reciprocity Method. Moreover, all terms of the matrix of coefficients are reduced to boundary integral expressions provided the particular solution trial functions satisfy the static problem obtained after discarding both non-homogeneous and time derivative terms from the governing equation.

1. Introduction

Hybrid-Trefftz elements are variants of the hybrid finite elements with the essential trait that the domain approximation basis is built using free-field solutions of the differential equation governing the problem [1]. Similar to boundary elements, the use of Trefftz-compliant trial functions reduces all terms of the solving system to boundary integral expressions. However, unlike conventional boundary elements, regular solutions of the governing equation are used to construct the trial basis (unless points of singularity are expected to occur) and the solving system is generally sparse and Hermitian.

Hybrid-Trefftz elements were first introduced by Stein in 1973 [2], but their widespread use and mathematical foundations are due to the pioneering work of Herrera (e.g. [3]) and Jirousek (e.g. [4]) through the 1980s and the 1990s. More recent contributions are due, for instance, to Qin (e.g. [5,6]) and Freitas (e.g. [7,8]). A recent monograph on the topic, including the discussion of relevant implementation issues and algorithms is also due to Qin [9].

As compared to conventional (conforming displacement) finite elements, hybrid-Trefftz elements feature trial bases selected from the free-field solution space of the differential equation governing the problem (the Trefftz-compliance condition). Consequently, the (hierarchical) approximation bases are tailored for each problem that is being solved. The considerable richness in physical information enables Trefftz-compliant trial bases to model accurately problems that are notoriously difficult to solve using conventional elements as, for instance, those involving highly oscillating or high gradient solutions, nearly incompressible constituents or grossly distorted meshes. The use of non-conforming meshes with super-sized finite elements is also affordable [8]. Another key advantage of hybrid-Trefftz finite elements is that the construction of the solving system does not involve summation of coefficients over neighbouring elements. This feature adds considerable flexibility to the definition of the model, as different orders of p-refinement can be defined on each finite element and essential boundary. The price to be paid for securing these advantages is having to deal with trial functions that are numerically heavier than their conventional counterpart and do not admit, in general, analytic integration. Moreover, Trefftz elements may experience additional difficulties when dealing with non-homogeneous boundary value problems, as discussed at large in this work.

Hybrid-Trefftz elements come in two flavors, which are coined here using the computational mechanics terminology as

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displacement model and stress model. The difference between the alternative models is mainly related to the boundary condition that is considered essential and thus explicitly enforced in the formulation (e.g. [10]). The displacement compatibility is enforced on the Dirichlet and inter-element boundaries in the displacement model. Conversely, in the stress model, the stress equilibrium is enforced on the Neumann and inter-element boundaries. Depending on the specific interest of the analyst, one model or another can be adopted. For instance, for limit state analyses in Structural Mechanics, the stress field is the primary objective, supporting the use of the stress model. Conversely, in thermic analyses, the temperature field may be of more interest and thus the temperature (i.e. equivalent to the displacement) model should be adopted.

When the problem to be solved is not homogeneous, the preservation of the above properties requires that an exact particular solution is found. Depending on the source (i.e. non-homogeneous) term, such closed form solution may or may not be available. For instance, analytic particular solutions were reported for elastostatic beams subjected to their own weight and for hollow cylinders subjected to axisymmetric temperature fields with logarithmic variations [9]. However, exact particular solutions are not easy to find, in general, for transient dynamic problems where the source terms are defined by the initial conditions of the current time step [11]. When this is the case, the Trefftz-compliant trial basis, constructed using functions that satisfy the homogeneous form of the governing equations, is generally unable to model correctly the particular solution since it lays outside of its span.

To overcome this issue, two main alternatives exist.

One is to evaluate the particular solution as the internal product of the source term with the fundamental solution of the differential operator present in the governing equation. This approach, while certainly viable, involves domain integrations of the (singular) fundamental solutions, a drawback that boundary methods typically tend to avoid.

The other mainstream alternative is the Dual Reciprocity Method (DRM). In the DRM, the particular solution is approximated using some additional trial functions, while the approximation of the complementary solution is handled by the Trefftz-compliant basis. However, if the additional trial functions were simply added to the Trefftz-compliant basis, they would cause, in general, domain integral terms to emerge in the matrix of coefficients, since they do not comply with the Trefftz condition. To avoid handling domain integrals, the DRM does not couple the particular and complementary solution bases. Instead, the particular solution is approximated first, using a two step process. In the first step, an approximation of the source function is obtained, typically using radial basis functions and domain collocation. In the second step, the differential equation is solved analytically, having the *approximation* of the source function as the non-homogeneous term [12]. Besides the inherent disadvantages of the collocation process (e.g. fully populated, non-symmetric and rather ill-conditioned systems), cumbersome expressions are generally obtained for the particular solution trial functions, which may also feature points of singularity, especially for hyperbolic differential operators [13].

In the finite element context, the DRM was applied by Qin and his co-workers to the solution of the Poisson's equation and to two-dimensional thermoelastic problems in [9], and to three-dimensional elastostatic and thermoelastic problems in [14] and [15], respectively. Radial basis functions were used to construct the source function approximation in all cases. An alternative approach aimed at avoiding the cumbersome expressions of the resulting particular solution trial functions was recently presented by Moldovan and Radu [16]. As opposed to other DRM variants, the technique uses the same trial functions for both source function and particular solution approximations. These functions have simple expressions and need not be singular, unless a singular particular solution is physically justified. The approximation is shown to be convergent and robust to mesh distortion.

Once an approximation of the particular solution is found, the complementary solution can be obtained using any boundary method. Hybrid-Trefftz finite elements are used by both Qin and Moldovan in the cited papers.

In a recent paper [11], we presented an alternative to the DRM for the solution of non-homogeneous hyperbolic boundary value problems. Cast in the framework of the displacement model of the hybrid-Trefftz finite element, the typifying feature of this approach is that the particular and complementary solution trial functions are merged together in the same trial basis. Consequently, the solution is obtained in a single step, through the solution of a sparse and Hermitian solving system, since no collocation is required. It is shown that a suitable choice of the particular solution trial basis is able to cancel out most of the domain integral terms in the matrix of coefficients, and those still remaining are numerically easy to evaluate. Moreover, the coupling of the particular and complementary trial functions allow the two parts of the basis to combine freely and compensate for each other's weaknesses. Unlike the DRM, however, the coupling of the particular and complementary solution bases means that the whole formulation must be established anew when the technique used to obtain the complementary solution is changed.

The procedure proposed in Ref. [11] is thus extended here in the context of the stress model of the hybrid-Trefftz finite element. Besides the advantages inherent to the use of the stress model in applications where the quality of the stress/flux field is more relevant to the analyst, this extension is justified by the additional advantages that the stress model features as compared to the displacement model presented in Ref. [11]. Indeed, the use of the stress model cancels out *all* domain integral terms in the matrix of the coefficients and one (of the two) domain integral terms on the right-hand side of the finite element solving system. Consequently, the stress model is simpler to implement, while still securing the key advantages of the displacement model, as discussed above.

The paper opens with the general definition of the hyperbolic boundary value problem that is tackled in the paper. The procedure for the discretization in time of the governing equations is succinctly described, since it is presented at length in Ref. [11]. The stress model of the hybrid-Trefftz finite element is formulated next, for the general hyperbolic problem. Finally, the application of the methodology is illustrated for a two-dimensional hyperbolic boundary value problem with analytic solution, in order to assess its accuracy and robustness.

2. Definition of the problem

Hyperbolic initial boundary value problems occur in many branches of engineering, like Acoustics, Electromagnetism and Structural Dynamics. Since Structural Dynamics tend to be the source of some of the most general hyperbolic problems and is an area where the use of stress models is highly justified, the following presentation uses the Structural Dynamics terminology in order to define the problem and coin the respective weights. However, the applicability of this approach is not limited in any way to this area and the applications presented in Section 5 are designed to reflect that.

A general hyperbolic initial boundary value problem defined over the domain V presented in Fig. 1 is typically described by the equilibrium, compatibility and constitutive equations,