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A serendipity triangular patch for evaluating weakly singular boundary integrals

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ABSTRACT

Element subdivision is the most widely used method for the numerical evaluation of weakly singular integrals in three-dimensional boundary element analyses. In the traditional subdivision method, the sub-elements, which are called patches in this paper, are obtained by simply connecting the singular point with each vertex of the element. Patches with large angles at the source point may be produced and thus, a large number of Gaussian quadrature points are needed to achieve acceptable accuracy. In this paper, a serendipity triangular patch with four-node is presented to solve the problem. Case studies have been made to investigate the effect of the location of the middle node of the serendipity patch on accuracy, and an optimal location is determined. Moreover, theoretical analysis validating the optimal location is also given with a new form of polar coordinate transformation. Numerical examples are presented to compare the new patch with the conventional linear patch with respect to both accuracy and efficiency. In all cases, the results are encouraging.

1. Introduction

Accurate and efficient evaluation of weakly singular integrals arising in the boundary integral equations (BIEs) is of crucial importance for successful implementation of the boundary element method (BEM) [1–6]. Much effort has been made to remove the weakly singular integrals appearing in BIEs. And there are many ways of evaluating singular integrals mentioned in the boundary element literature. These approaches include integral simplification [7], element subdivision [8] and coordinate transformation [9], with each method having its advantages and disadvantages. Element subdivision is one of the most widely used methods for the numerical evaluation of weakly singular integrals. Many different element subdivision methods have been proposed. Klees has proposed a subdivision method and the sub-elements which are also called patches are obtained by simply connecting the singular point with each vertex of the element [8]. Zhang et al. have used the subdivision method coupled with a new coordinate transformation which is denoted as (α-β) transformation to remove singularities [10] and further developed an adaptive element subdivision method named Quad-tree subdivision [11]. Obviously, all the above element subdivision methods may produce patches in “bad” shapes. Moreover, polar coordinate transformation is a powerful and useful tool to evaluate weakly singular integral in boundary element. It converses the surface integral into a double integral in radial and angular directions. Many works have been done on dealing with the singularity in the radial direction; however, numerical integration on the angular direction still deserves more attention. In fact, after singularity cancelation or subtraction, although the integrand may behave very well in the radial direction, its behavior in the angular direction would be much worst, so too many quadrature points are needed. Especially when the source point lies close to the boundary of the element, one can clearly observe near singularity of the integrand in the angular direction. Similar problems have been considered in work about the nearly singular BEM integrals. Effective methods along this line are the subdivision method [12], the Hayami transformation [13], the sigmoidal transformation [14], the conformal transformation [15], the variable transformation [16], etc.

In this paper, Sphere subdivision method proposed by Zhang is used [17], and based on this method; first a serendipity triangular patch which is obtained by the element subdivision is introduced to overcome the problem of the integral in the angular direction. One edge of the patch is replaced by quadratic curve, the distance between the middle node of the quadratic curve and the source point is equal to the length of radius of the sphere which is centered at the source point. Then the polar coordinate system with

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new form is used in this patch. This system is very similar to the conventional polar system, but its implementation is simpler than the conventional polar system and also performs efficiently. Using the coordinate transformation, the integrals with weakly singularities can be accurately calculated. Furthermore, in order to investigate the effect of the location of the middle node on the computational accuracy, its location is changed along the direction of the source point to the middle node step by step. Through theoretical analysis and numerical experiment, the location with the highest accuracy has been found in this paper. With our method, the weakly singular boundary integrals in the regular or irregular elements can be accurately and effectively calculated. And our method can be also applied to the patch with large angles at the source point. Numerical examples are presented to validate the proposed method. Results demonstrate the accuracy and efficiency of our method.

This paper is organized as follows. Detailed description of the serendipity triangular patch and the coordinate system with new form are described in Section 2. In Section 3, the effect of the middle node's location in the serendipity triangular patch is introduced. Numerical examples are given in Section 4. The paper ends with conclusions in Section 5.

2. Four-nodes serendipity triangular patch

To achieve the best balance between accuracy and efficiency, it is desirable that subdivided patches closer to the source point have relatively smaller sizes. To guarantee this, we use a sequence of spheres centered at the source point with decreasing radius to cut the element, recursively. And a serendipity triangular patch with four-nodes is obtained through the element subdivision.

The serendipity triangular patch is as shown in Fig. 1, the following symbols are defined:

0—the source point;

3—the middle node;

0, 1, 2, 3—serendipity patch node;

In the serendipity patch, the distance between point 0 and point 3 is equal to the length of radius of the sphere which is centered at the source point. And the length of radius we can obtain in [17]. For the patches containing source point, the coordinate transformation is used to eliminate the singularities.

Considering the weakly singular integral over a patch as shown in Fig. 2, the following boundary integral can be represented as

$$I(y) = \int_s \frac{f(y, r)}{r}\phi(x)dS \tag{1}$$

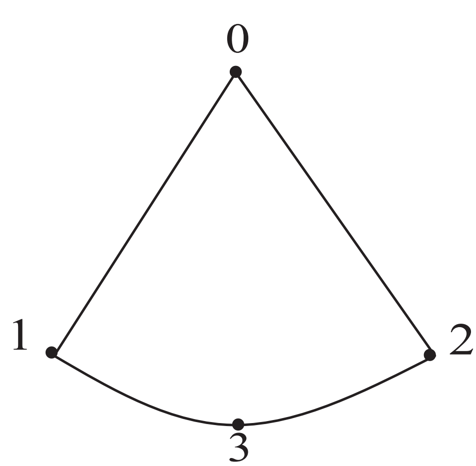

Fig. 1. Four-nodes serendipity triangular patch.

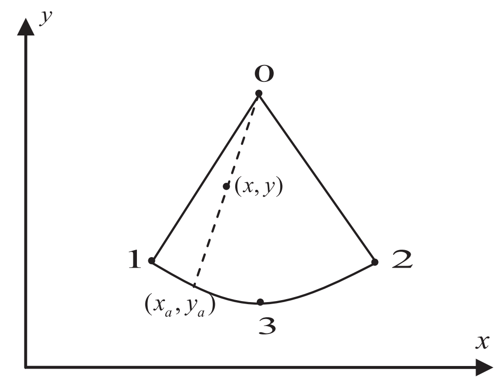

Fig. 2. The coordinate transformation of serendipity triangular patch.

where y and x are referred to as the source point and the field point, respectively, r is the distance between y and x, f is a well-behaved function, and $\phi(x)$ is a shape function.

For this patch, to construct the local (ρ, θ) system, the following mapping is used:

$$\begin{cases} x_a = N_0x_1 + N_1x_2 + N_2x_3 \\ y_a = N_0y_1 + N_1y_2 + N_2y_3 \end{cases} \tag{2a}$$

In the Eq. (2a), N_0, N_1 and N_2 are the shape functions of the quadratic curve.

$$\begin{aligned} N_0 &= \frac{1}{2}\theta(\theta - 1) \\ N_2 &= (1 + \theta)(1 - \theta) \\ N_1 &= \frac{1}{2}\theta(\theta + 1) \end{aligned} \tag{2b}$$

$$\begin{cases} x = x_0 + (x_a - x_0)\rho \\ y = y_0 + (y_a - y_0)\rho \end{cases} \quad \rho \in [0, 1], \theta \in [-1, 1] \tag{2c}$$

Combining (Eqs. (2a), 2(b) and 2c), the coordinate transformation can be written as:

$$\begin{cases} x = x_0 + \left([N_0x_1 + N_2x_3 + N_1x_2] - x_0\right)\rho \\ y = y_0 + \left([N_0y_1 + N_2y_3 + N_1y_2] - y_0\right)\rho \end{cases} \tag{3}$$

Then the integral I can be written as

$$I = \int_{-1}^{1}\int_0^1 \frac{f(y, r)}{r}Jb(\rho, \theta)\phi d\rho d\theta \tag{4}$$

where $Jb(\rho,\theta)$ is the Jacobian of the transformation from the x–y system to the ρ–θ system,

$$Jb(\rho, \theta) = \rho\left[(x_a - x_0)\frac{\partial y_a}{\partial \theta} - (y_a - y_0)\frac{\partial x_a}{\partial \theta}\right] \tag{5}$$

From Eqs. (2) to (5), it can be noted that the new coordinate system is much simpler to implement than the conventional polar coordinate system [9]. This is due to the fact that ρ and θ are constrained to the interval [0, 1] and [−1, 1] in this triangle, thus there is no need to calculate their spans. So our method may be computationally more efficient.

3. The effect of the middle node's location

3.1. Changing the middle node's location

In this section, we will investigate the effect of the middle node's location on the computational accuracy. As shown in Fig. 3,