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Computing eigenmodes of elliptic operators using increasingly flat radial basis functions

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ABSTRACT

Solving multi-dimensional eigenmodes problem for elliptic operator using radial basis functions (RBFs) was proposed by Platte and Driscoll (2004) [14]. They convert the eigenmodes problem to an eigenpairs problem of a finite dimensional matrix. We formulate an approach based on using finite order interpolating polynomials as eigenfunctions for eigenmodes problem. We prove that, under some simple conditions on the RBFs, two approaches converge when increasingly flat BRFs are being used. These results are supported by numerical examples.

1. Introduction

Multiquadric (MQ) collocation method, first pioneered by Kansa [10], is a highly efficient numerical method for solving partial differential equations (PDEs). Platte and Driscoll [14] also discussed using Radial Basis Functions (RBFs) to compute eigenmodes of elliptic operators. They concluded from numerical results that RBFs method is superior to basic finite-element methods for computing eigenmodes of Laplace operator in two dimensions.

To be precise, they studied the following eigenmodes problem of elliptic operators. Given a linear elliptic second-order partial differential operator $\mathfrak{L}$ and a bounded region Ω in $\mathbb{R}^n$ with boundary $\partial\Omega$, they sought eigenpairs $(\lambda, u) \in (\mathbb{C}, C(\overline{\Omega}))$ satisfying

$$\begin{aligned} &\mathfrak{L}u + \lambda u = 0, \quad \text{in } \Omega, \quad \text{and} \\ &\quad \mathfrak{L}_{\mathfrak{B}} u = 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\mathfrak{L}_{\mathfrak{B}}$ is a linear boundary operator of the form

$$\mathfrak{L}_{\mathfrak{B}} u = au + b(\mathbf{n} \cdot \nabla u). \tag{2}$$

Here, a and b are given constants and $\mathbf{n}$ is the unit outward normal vector defined on the boundary. Firstly, they used an interpolating RBF approximation of an eigenfunction of (1), then they approximated the operator $\mathfrak{L}$ by a matrix that incorporates the boundary conditions. Now, the above eigenvalue problem has been replaced with a finite-dimensional eigenvalue problem. Finally, standard techniques to find the eigenvalues and eigenvectors of this matrix are used.

The behaviors of the solutions in the limit of nearly flat RBFs have been studied for both interpolation problems and partial differential equations. In the study of several types of RBFs interpolation, such as MQ, Driscoll and Fornberg [4] proved that, in the 1-D case, the interpolant converges to the Lagrange interpolating polynomial as the basis functions become increasingly flat. For multivariate interpolation problem, Larsson and Fornberg [11] also proved that the interpolating RBF approximation converges to an interpolating polynomial, provided suitable constraints on the given data points.

For solving 1-D Poisson's equation, Chen et al. [3] proved that the approximation by using increasingly flat RBFs in the method of fundamental solutions (MFS) to the given Poisson's equation reduces to simply interpolation problem and solution converges in the sense of Lagrange interpolating polynomial. It is the purpose of this paper to show that similar results can be obtained for solving eigenvalue problem for Laplace operator (1) on a multi-dimensional region. We show that the final matrix associated to the finite-dimensional eigenvalue problem using increasingly flat RBFs converges entry by entry to the matrix obtained by using interpolating polynomial as an eigenfunction of (1). This interpolating polynomial is obtained from the limiting interpolant using increasingly flat BRFs as in [4]. Note that, in one space dimension, the interpolant is the Lagrange interpolation polynomial. Therefore, the eigenvalues obtained from two approaches converge. In other words, we show that solving eigenvalue problem for Laplace operator on a multi-dimensional region

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with increasingly flat RBFs has similar asymptotic behaviors as solving interpolation problem.

In general, a RBFs approach scheme is to write the approximated solution as a finite expansion of RBFs, then solve for the coefficient set of the expansion using collocation method. We prove that, using coefficients obtained from an interpolation problem, the expansion with respect to the Laplacian of the RBFs converges to the Laplacian of the interpolating polynomial as the RBFs become increasingly flat. This crucial proposition leads to the main results stated above.

The paper is organized as follows. In Section 2, we describe the interpolation process using RBFs. We review the main results of Larsson and Fornberg [11] regarding finding the interpolant using increasingly flat RBFs in Section 3. The main proposition that leads to main theorem is also given in Section 3. In Section 4 we provide two approaches to solve the eigenmodes problem for Laplace operator with homogeneous Dirichlet boundary condition on a multi-dimensional region. And we show the main theorem that these two approaches converge when increasingly flat RBFs are applied. Some numerical results are provided in Section 5. Concluding remarks are given in Section 6.

2. Radial basis function collocation method

2.1. Radial basis functions

A radial function is a multivariate function Φ such that

$$\Phi : \mathbb{R}^d \to \mathbb{R} \quad \text{in the sense that} \quad \Phi(x_1,\dots,x_d) \to \phi(\|(x_1,\dots,x_d)\|_2). \tag{3}$$

Here the Euclidean norm of $\mathbf{x} = (x_1,\dots,x_d)$ is defined as

$$\|\mathbf{x}\| = \sqrt{\sum_{i=1}^{d} x_i^2} = r. \tag{4}$$

RBF collocation method is an "element-free", or a "meshless", technique for generating data-dependent spaces of multivariate functions. The spaces are spanned by shifted and scaled radial functions. The shifting is accomplished by using a set of scattered *centers*, $\mathbf{y}_1,\dots,\mathbf{y}_N$ in $\mathbb{R}^d$, sometimes called *basis points*, as the origin of the RBFs. Reconstruction of functions is then made by trial functions which are linear combinations:

$$\tilde{u}(\mathbf{x}) := \sum_{j=1}^{N} \alpha_j \Phi(\mathbf{x},\mathbf{y}_j) = \sum_{j=1}^{N} \alpha_j \phi(\|\mathbf{x}-\mathbf{y}_j\|). \tag{5}$$

Some of the commonly used RBFs are given in Table 1. All of these RBFs can be scaled by a shape parameter c, or $\epsilon = 1/c$, that controls the flatness (or steepness) of the RBF. This is done in such a way that $\phi(r)$ is replaced by $\phi(\epsilon r)$, or $\phi(r/c)$. For example, the MQ is scaled as $\phi = \sqrt{(\epsilon r)^2+1}$, or $\phi = \sqrt{(r/c)^2+1}$. The effect of the scaling is that as ϵ gets smaller, or c gets larger, the RBF becomes flatter. Further detail on RBF can be found in the following excellent books [2,5–7,16].

Table 1
Some examples of radial basis functions.

Infinitely smooth RBFs	
RBF	$\phi(r)$
Gaussian (GA)	e^{-r^2}
Multiquadric (MQ)	$\sqrt{r^2+1}$
Inverse Multiquadric (IMQ)	$1/\sqrt{r^2+1}$
Inverse Quadratic (IQ)	$1/(r^2+1)$

2.2. Interpolation

A typical interpolation problem has the following form: given scattered data points $\{\mathbf{x}_i\}_{i=1}^N$, and data values $\{u_i\}_{i=1}^N$, find an interpolant

$$\tilde{u}(\mathbf{x}) = \sum_{j=1}^{N} \alpha_j \phi\left(\|\mathbf{x}-\mathbf{y}_j\|\right), \tag{6}$$

where $\{\mathbf{y}_j\}_{j=1}^N$ are the center of the radial basis functions. The interpolation at collocation points gives

$$\tilde{u}(\mathbf{x}_i) \equiv \sum_{j=1}^{N} \alpha_j \phi\left(\|\mathbf{x}_i-\mathbf{y}_j\|\right) = u_i, \quad i = 1,\dots,N. \tag{7}$$

This summarizes in a system of equations for the unknown coefficients α_j,

$$\mathcal{A}\mathbf{\Pi} = \mathbf{u}, \tag{8}$$

where $\mathcal{A}$ is an $N \times N$ matrix with elements $\mathcal{A}_{ij} = \phi\left(\|\mathbf{x}_i-\mathbf{y}_j\|\right)$, $\mathbf{\Pi} = [\alpha_1\ \alpha_2\ \cdots\ \alpha_N]^T$, and $\mathbf{u} = [u_1\ u_2\ \cdots\ u_N]^T$. The non-singularity of such system, with distinct centers $\{\mathbf{y}_j\}_{j=1}^N$, was established in the 1930s by Bochner [1] and Schoenberg [15], and in the 1980s by Micchelli [13], see also a review article [8]. In fact, we can see $\mathbf{\Pi}$ as a function from $\mathbb{R}^N$ to $\mathbb{R}^N$, i.e. $\mathbf{\Pi} : \mathbb{R}^N \to \mathbb{R}^N$ and $\mathbf{\Pi}(\mathbf{u}) = \mathcal{A}^{-1}(\mathbf{u})$ if $\mathcal{A}^{-1}$ exists.

We could write (6) as

$$\tilde{u}(\mathbf{x},\epsilon) = \sum_{j=1}^{N} \alpha_j \phi\left(\|\mathbf{x}-\mathbf{y}_j\|,\epsilon\right), \tag{9}$$

where we have explicitly brought out the role of ϵ $(=1/c)$ in the approximation. Driscoll and Fornberg [4] pointed out that the RBFs presented in Table 1 belong to a class of infinitely smooth RBFs that can be expanded into a power series

$$\phi(r_j,\epsilon) = \phi\left(\|\mathbf{x}-\mathbf{y}_j\|,\epsilon\right) = a_0 + a_1(\epsilon r_j)^2 + a_2(\epsilon r_j)^4 + \cdots = \sum_{i=0}^{\infty} a_i(\epsilon r_j)^{2i}, \tag{10}$$

with the coefficients given in Table 2.

3. Definitions and interpolants with increasingly flat RBFs

This section contains definitions for multi-index notations, we also review some properties of polynomial interpolation and theorems about interpolants with increasingly flat RBFs in [11]. By taking Laplacian of the classic RBFs, we obtain new families of BRFs, we call them Laplacian RBFs. We prove similar results as in [11] about interpolant in the limit $\epsilon \to 0$ with these Laplacian RBFs. That is we show that, using coefficients obtained from an interpolation problem, the expansion with respect to the Laplacian RBFs converges to the Laplacian of the interpolating polynomial as the RBFs become increasingly flat.

Table 2
Expansion coefficients for infinitely smooth RBFs.

RBF	Coefficients	
GA	$a_i = \frac{(-1)^i}{i!}$,	$i = 0, \dots$
MQ	$a_0 = 1, a_i = \frac{(-1)^{i+1}}{2i}\prod_{k=1}^{i-1}\frac{2k-1}{2k}$,	$i = 1, \dots$
IMQ	$a_0 = 1, a_i = (-1)^i\prod_{k=1}^{i}\frac{2k-1}{2k}$,	$i = 1, \dots$
IQ	$a_i = (-1)^i$,	$i = 0, \dots$