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A domain decomposition based method for two-dimensional linear elastic fractures

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ABSTRACT

In this study, the two-dimensional physical domain containing cracks is divided into several non-overlapping parts: rectangular crack-tip regions around crack tips and the outer region without any crack tip. In each crack-tip region the displacement is approximated with Williams' series; while in the outer region it is approximated with numerical manifold interpolation. In order to balance accuracy and efficiency in solution, a transitional zone encompassing each crack-tip region is locally refined with a structured mesh. To avoid singular integration over a crack-tip region, the potential energy over every crack-tip region is transformed into the boundary integration. Three different methods to enforce compatibility on interfaces are compared, concluding the Lagrange multiplier method is superior over the other two.

1. Introduction

Fracture problems have always been popular topics in academic communities. During the early stage of the history of fracture mechanics, a large number of analytical solutions for fracture problems with very regular domains were gained. Given that the models of practical fracture problems are mostly irregular, numerical methods are largely employed in research and application nowadays. Numerical solution for linear elastic fractures is one of the most important aspects of theory and application of different numerical methods. Numerical methods for this problem include extended finite element method (XFEM) [1,2], element-free Galerkin method (EFGM) [3], generalized finite element method [4,5], numerical manifold method (NMM) [6,7], cracking particles methods [8,9], other meshfree methods [10–13], finite element methods based on remeshing [14–17], extended isogeometric analysis method [18,19], phantom node method [20], to name just a few.

In the numerical analysis of linear elastic fractures, the most fundamental and crucial step is to determine the stress intensity factor (SIF) and crack tip field. The stress at a crack tip is infinite. It is greatly difficult to promote accuracy of SIF simply by refinement of mesh due to singularity. In order to improve accuracy, Fleming [21] et al. enriched the approximations of element free Galerkin method with controlling bases of asymptotic crack tip field. Many others also incorporated these bases into local approximations in the solution of linear elastic fractures, such as extended element method [1] and numerical manifold method [22,23]. Direct incorporation of controlling bases of asymptotic crack tip field into local approximations is succinct and easy to implement, and the precision of solution is incredibly enhanced. Unfortunately, Fries and Belytschko [24] pointed out that this direct enrichment leaded to singular stiffness matrix in extended finite element methods. In numerical manifold method the phenomena were also observed.

Williams' series [25] are the analytical solutions of asymptotic crack tip field. Forthright enrichments with a few items of Williams' series in XFEM, EFGM and NMM brought about incredible accuracy. Therefore, one can naturally think of approximating displacement, strain and stress in the region around each crack tip by multiple terms of Williams' series. The singular stiffness matrix caused by direct enrichment of local approximations with a finite number of items of Williams' series in XFEM and NMM is a burdensome problem. A feasible treatment of this problem is to set a super element around each crack tip. One can set a super element encompassing the crack tip, in which the displacement is approximated in a totally different way from that of ordinary elements. Some treatments are needed to ensure compatibility of displacement on the outside boundary of the super element. The

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hybrid crack element [26] proposed by Tong et al. is an archetype of this sort of element.

Tong [26] et al. approximated the displacement within the hybrid crack element with complex series, while Karihaloo and Xiao [27] with Williams' series. Karihaloo and Xiao [27] reformulated the hybrid crack element, and evaluated the coefficients of Williams' series with this method. Then they solved a variety of problems by hybrid crack element [28,29]. Su and Feng [30] also studied this method. Since elements of FEM have to accommodate to crack surfaces, the preprocessing of problems with complicated domains or cracks is very challenging. While the crack can extend through elements in XFEM, the mesh of XFEM does not have to adapt to the crack surface. Thus XFEM is more suitable for fracture problems than FEM. Based on this, Xiao and Karihaloo [31] combined the hybrid crack element with XFEM to establish a novel method, and solved linear elastic fracture problems with the new method. Later, Passieux [32] et al. proposed a multigrid XFEM to directly determine the SIF.

In all the applications of hybrid crack element mentioned above, the functional of energy of the hybrid crack element was formulated based on potential energy or generalized potential energy. In fact, the functional of the hybrid crack element can also be constituted with complementary or generalized complementary energy. Long [33] proposed piecewise generalized variational principles in 1981. Long [34] et al. divided the domain for an linear elastic fracture problem into several non-overlapping regions, and approximated displacements with FEM interpolation in the non-crack-tip region on which the potential energy is defined, and stresses with Williams' series in the crack-tip region on which the complementary energy is defined. According to one of the piecewise variational principles [33], Long [34] et al. obtained the energy functional on the entire problem domain based on complementary energy and potential energy. By using this method, Long et al. got extremely accurate SIFs. Zhuang [35] et al. also utilized the piecewise variational principle [33], but in the non-crack-tip region replaced FEM interpolation with meshless radial point interpolation, and developed a new method for the evaluation of the linear elastic crack tip field.

NMM [6] is a reasonable choice for the numerical solution of problems with discontinuities such as cracks. One can simulate the discontinuous displacements across crack surfaces by NMM in an extraordinarily concise manner. The treatment of multiple and intersecting cracks with XFEM is cumbersome, while NMM can handle these very easily. No extra techniques are needed to form the approximation of the displacement in the case of complex cracks. The only necessary step is to cut mathematical patches with crack surfaces. NMM has been utilized to deal with fracture problems in many cases successfully [22,23,36–42]. Hence, the advantage of NMM to model crack surfaces will be taken in this study, together with highly precise crack tip field represented by Williams' series, in order to formulate decomposition methods for linear elastic fractures.

The outline of the paper is as follow. Section 2 is a brief illustration of NMM. The representation of discontinuous displacements across crack surfaces is underlined. The basic idea of partition of unity based refinement method is briefed in Section 3. The refinement of meshes for practical problems is also stated. Section 4 is the illustration of the division of domains for linear elastic fracture problems, approximation of displacement and energy of each region. In addition, potential energy in the form of integral over the crack-tip region is transformed into integral only along boundaries. Thus the treatment of singularity at the crack tip is circumvented. Three domain decomposition methods are elaborated in Sections 5–7 respectively. We employ Lagrange multiplier in Section 5 to enforce compatible displacements across interfaces. The dual mortar method for the determination of interpolation function for the Lagrange multiplier is stated. Several factors which can affect the precision of the solution are studied and multiple problems with complex cracks are solved to validate the proposed method. The penalty method is adopted in Section 6 for a new domain decomposition method. In Section 7 another domain decomposition method according to a piecewise variational principle proposed by Long [33] is established. Section 8 is the conclusion of the paper.

2. Brief illustration of NMM

The numerical manifold method (NMM) proposed by Shi [6] is powerful in modeling static and dynamic failure of materials. It was elaborated and reviewed in several papers [43–45]. The basic theory of NMM will be briefly illustrated in the following, and its advantage in modeling discontinuities such as fractures will be underlined.

2.1. Mathematical cover, physical cover and manifold elements

NMM introduces two cover systems, the mathematical cover (MC) and physical cover (PC). Mathematical cover consists of a series of connected regions. Each one of these connected regions is called a mathematical patch, which is usually a simply connected domain. The union of all mathematical patches is the mathematical cover. The mathematical cover does not have to accommodate to boundaries as in FEM, as long as the mathematical cover can contain the physical domain entirely, which is the only requirement for mathematical covers. If the problem domain is denoted by Ω, and each mathematical patch by M_i ($i=1, 2, \ldots m$, m is the number of mathematical patches), the relation between them can be expressed as

$$\Omega \subseteq \cup_{i=1}^{m} M_i \quad (1)$$

Every mathematical patch is cut into several connected regions by the boundaries and the discontinuous interfaces of the problem domain. Having been cut, some of these regions are totally contained by the problem domain, while others are outside the problem domain and to be discarded. Each of the remaining regions is called a physical patch. Each mathematical patch may contain one or more physical patches. A mathematical patch is split into more than one physical patch if and only if it is cut entirely by discontinuities. Otherwise, only one physical patch is formed which is less than or exactly equal to the mathematical patch. A typical physical patch is denoted by P_j ($j=1, 2, \ldots p$, p is the number of physical patches). The set of all the physical patches is the physical cover (PC), which covers the problem domain exactly, namely,

$$\Omega = \cup_{j=1}^{p} P_j \quad (2)$$

Because the mathematical patches overlap, the physical patches also overlap. Thus, each physical patch P_j may contain boundaries of other physical patches. All the boundaries of other physical patches inside P_j divide P_j into several connected regions. Each of these regions is called a manifold element. A manifold element can be contained by one or more physical patches, which together decide the approximation of displacement inside the manifold element. Distinct manifold elements are disjointed, and all the manifold elements partition the problem domain exactly. A typical manifold element is denoted as E_k ($k=1,2\ldots e$, e is the number of manifold elements). The relation between manifold elements and physical domain is expressed as

$$\Omega = \cup_{k=1}^{e} E_k \quad (3)$$

Manifold elements serve as the basic units for integration of weak form.