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Slope limiters for radial basis functions applied to conservation laws with discontinuous flux function ☆

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ABSTRACT

We present slope limiters in meshless radial basis functions for solving nonlinear equations of conservation laws with flux function that depends on discontinuous coefficients. The method is based on the local collocation formulation and does not require either generation of a grid or evaluation of an integral. Upwind techniques are used to allocate collocation points within the characteristic solutions and different slope limiter functions are investigated. The main advantages of this approach are neither mesh generations nor Riemann problem solvers are required during the solution process. Numerical results are shown for several test examples including models on vehicular traffic and two-phase flows. The main focus is to examine the performance of the proposed meshless method for shock-capturing property in conservation laws with discontinuous flux function. The obtained results demonstrate its ability to capture the main solution features.

1. Introduction

Nonlinear conservation laws with discontinuous flux function occur in many physical applications, for example in porous media flows [10], sedimentation phenomena [8], resonant models [12] and vehicular traffic [18] among others. In general, the problem statement in this class of applications consists of numerically solving the Cauchy problem associated with the following scalar conservation laws:

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}f(k(x),u)=0,\quad x\in\mathbb{R},\quad t>0,$$
$$u(0,x)=u_0(x),\quad x\in\mathbb{R}, \tag{1}$$

where $u\in\mathbb{R}$ is the scalar unknown, the flux function $f(k(x),u):\mathbb{R}\longrightarrow\mathbb{R}$ is nonlinear and $k(x)$ is a given function that can depend on time variable as well. We assume that the Jacobian $f_u(k(x),u)=\partial f(k(x),u)/\partial u$ is diagonalizable with real eigenvalues. To illustrate the numerical techniques discussed in this paper, we use a multiplicative form of the flux function defined as

$$f(k(x),u)=k(x)g(u),\qquad k(x)=\begin{cases}k_L, & \text{if } x<x_0,\\ k_R, & \text{if } x>x_0,\end{cases} \tag{2}$$

where x_0 is the location of the interface, k_l and k_R are given constants with $k_L\neq k_R$. Note that most practical applications of these problems cannot be solved analytically and hence require numerical methods to approximate their solutions. One of the main difficulties in the analysis of conservation laws (1) and (2) is the correct definition of a solution. It is well known that after a finite time, the problem (1) and (2) does not in general possess a continuous solution even if the initial data $u_0(x)$ is sufficiently smooth. Hence a solution of (1) and (2) has to be understood in the weak sense. Moreover, among the computational difficulties that arose when approximating solutions of the problem (1) and (2) are numerical instability, poor shock and rarefaction resolutions, and even spurious numerical solutions.

Many numerical methods are available in the literature to solve conservation laws with discontinuous flux function. The most popular techniques are finite volume schemes which are based on exact or approximate solver such as Lax-Friedrich, Godunov and Engquist-Oscher methods among others. In fact, the Godunov-type methods use approximate Riemann solvers to compute the numerical flux instead of the exact Riemann solvers. These procedures are mathematically hard to treat and

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computationally demanding, particularly for flux functions changing the derivative signs in more than one interface or depending on discontinuous time-dependent coefficients. In addition, application of the Godunov method to solve conservation laws with discontinuous flux function requires a discretization of the discontinuous coefficients staggered with respect to that of the solution, compare [28,6] and further details are therein. In the framework of relaxation approximations of conservation laws, a class of Monotonic Upstream-Centered Schemes for Conservation Laws (MUSCL) methods has been applied in [25] to solve the problem (1) and (2). It has been shown in this reference that the relaxation system associated with the problem (1) and (2) regularizes the solution of the original problem but the numerical diffusion in the presented results is clearly noticeable.

Mesh-based techniques such as finite difference, finite element and finite volume methods have been widely used for solving partial differential equations. However, the accuracy of these methods is affected by the quality of the meshes, stabilization techniques and solution of Riemann problems, which hinders their applications to solving real problems with irregular domains and complex Riemann problems. Note that the introduction of artificial viscosity has been widely used to stabilize many finite element methods for conservation laws whereas the solution of Riemann problems is roughly approximated in many finite volume methods. Significant developments in meshless methods for solving linear and nonlinear partial differential equations have been achieved. For instance, the meshless local Petrov–Galerkin and local boundary integral equation methods were studied in [1,2]. These methods basically transformed the original problem into a local weak formulation and the shape functions were constructed from using the moving least-squares approximation to interpolate the solution variables. Meshless Radial Basis Functions (RBFs) have been subject to several studies and their applications to solve partial differential equations have also been covered in the literature. The RBF approximations, particularly the multiquadric basis functions, were first devised for scattered geographical data interpolation in [29,19]. A review on the application of RBF methods for scattered data interpolation can be found in [14]. Theoretical results for RBF have also been presented in [4,21] among others. These results include solvability, convergence and stability of the RBF interpolation in a general framework. Application of the RBF methods to steady and time-dependent partial differential equations has also been investigated, see for example [11,15]. Recently the RBF methods have also been used to solve hyperbolic systems of conservation laws such as the Euler system for gas dynamics and shallow water equations in [26,16]. The current paper devises a truly meshless RBF method for conservation laws with discontinuous flux function and its application to problems in vehicular traffic and two-phase flows. In order to reconstruct a numerical scheme that maintains the high-order accuracy away from discontinuities, while producing monotone results at discontinuities, we adapt techniques from slope limiters in mesh-based methods for conservation laws to the local meshless RBF method. The key idea is to benefit from the non-oscillatory character of the upwind collocation and to blend it with a local RBF method. The blending procedure is carried out by standard slope limiter functions. The results using the proposed RBF method with slope limiters are presented for several test problems. To the best of our knowledge, solving conservation laws with discontinuous flux function using these numerical tools is reported for the first time.

The remainder of the paper is organized as follows. In Section 2 we present the meshless radial basis function method for conservation laws. For simplicity of presentation we will consider the discontinuous flux function given by (2), but methods herein presented extend directly to other general flux functions and are also applicable to systems of conservation laws. Next, slope limiters in the framework of radial basis functions are formulated in Section 3. In Section 4 we extend the meshless radial basis function method to two-dimensional problems. Finally, Section 5 presents numerical results for various test examples on scalar conservations appeared in modeling vehicular traffic and two-phase flow in porous media. Conclusions are drawn in Section 6.

2. Radial basis functions for conservation laws

Let us assume that a nodal distribution of N distinct points x_j is used as a collocation in the computational domain and let $w_i(t)$ denotes the value of a generic function w at the collocation point x_i and time t. The main idea of the interpolation using local RBF is to interpolate the unknown function $w_i(t)$ by the expansion

$$w_i(t) \simeq \sum_{j \in I_{i,m}} \lambda_j(t)\varphi\left(|x_i - x_j|\right) + \sum_{k=1}^{M} \lambda_k p_k(x), \tag{3}$$

where $I_{i,m}$ is the local set containing the index i and indices of the neighboring points to the reference point x_i. In (3), $p_k(x)$ are exactly the polynomials spanning π_M which are polynomials of degree at most M and satisfying the constraints

$$\sum_{j \in I_{i,m}} \lambda_j p_k(x_j) = 0, \quad k = 1, 2, \ldots, M, \tag{4}$$

where λ_j's are the unknown coefficients to be calculated, $r_{ij} = |x_i - x_j|$ is the distance between the points x and x_j, and $\varphi(|x_i - x_j|)$ is the radial basis function. In the current study we consider the infinitely smooth multiquadrics radial basis function defined as

$$\varphi(r) = \sqrt{1 + \epsilon^2 r^2}, \tag{5}$$

where $\epsilon \neq 0$ is the shape parameter controlling the fitting of a smooth surface to the data. The lack of the mathematical theory makes it very difficult to choose a suitable value ϵ for the RBF methods. However, investigations in [14,5] have shown that the computational accuracy of the radial basis interpolation can be improved by varying the shape parameter with the selected radial function. In the present work we used the following selection [9]:

$$\epsilon = 0.8 \frac{\sqrt{n_s}}{d_m}, \tag{6}$$

with n_s being the cardinal number of the set $I_{i,m}$ and d_m denotes the smallest nodal distance in $I_{i,m}$. Note that other selections for the shape constant ϵ and other radial basis functions can be easily incorporated in our analysis without major conceptual modifications.

The selection of the set $I_{i,m}$ may depend on the problem under study and for purpose of this study upwinding techniques are adopted for the selection of the set $I_{i,m}$. Thus, for each collocation point x_i the associated set $I_{i,m}$ contains the index i and indices of the m nearest neighboring points to both sides of x_i. It is evident that for this selection the cardinal number of the set $I_{i,m}$ is given by $n_s = 2m + 1$. Using this selection of the set $I_{i,m}$ the expansion coefficients $\lambda_i(t)$ in (3) are obtained by solving the following linear system of $n_s \times n_s$ algebraic equations

$$\mathbf{B}^{[i]} \Lambda^{[i]} = \mathbf{w}^{[i]}, \tag{7}$$

where $\mathbf{B}^{[i]}$ is an $n_s \times n_s$ matrix with entries $\varphi(|x_i - x_j|)$, $\Lambda^{[i]}$ and $\mathbf{w}^{[i]}$ are n_s-valued vectors with entries λ_i and w_i, respectively. For many choices of radial basis functions φ, the interpolation matrix $\mathbf{B}^{[i]}$ in (7) is guaranteed to be nonsingular for any set of distinct points and the invertibility is therefore guaranteed, see for example [19,21]. Notice that adding the polynomial terms to the considered radial basis interpolation (3) is not generally required. However, this argument may not be applicable to other radial basis functions.

Partial derivatives of the interpolant (3) may be calculated in a straightforward manner. For instance, the temporal and spatial