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Stochastic spline fictitious boundary element method for modal analysis of plane elastic problems with random fields

Cheng Su [a,b], Zhongshan Qin [a], Xueming Fan [a,b,*]

[a] School of Civil Engineering and Transportation, South China University of Technology, Guangzhou 510640, PR China
[b] State Key Laboratory of Subtropical Building Science, South China University of Technology, Guangzhou 510640, PR China

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ABSTRACT

Mathematical formulation and computational implementation of the stochastic spline fictitious boundary element method (SFBEM) are presented for modal analysis of plane elastic problems with structural parameters modeled as random fields. Two sets of governing differential equations with respect to the means and deviations of displacement modes are derived by including the first order terms of deviations. These equations are in similar forms to those of deterministic plane elastostatic problems, and can be solved using deterministic elastostatic fundamental solutions, resulting in the means and covariances of the eigenvalues and mode shapes. For the effective treatment of the domain integrals involved in the deviation solution, the random fields considered are represented by Karhunen–Loeve (KL) expansion in conjunction with the Galerkin projection. Numerical examples indicate that the results of the present method are in good agreement with those from the Monte Carlo simulation (MCS) with small variations, and the present approach is more efficient than the perturbation stochastic finite element method (FEM) with the same KL expansion technique.

1. Introduction

Modal analysis of a structure is one of the major issues in structural design, and is the foundation for dynamic response analysis of the structure as well. In conventional structural design, modal analysis is generally based on deterministic structural models, in which the effects of the inherent randomness in structural systems have been completely neglected. However, it has been found that the structural uncertainties have a certain influence on the modal responses of structures [1–3]. Therefore, it is more appropriate to take into account these uncertainties at the modeling level in structural modal analysis.

In the field of computational stochastic mechanics, the perturbation method, due to its considerably high efficiency, is the most commonly used approach, which is mainly applied in the framework of finite element method (FEM) for its wide utilization in the deterministic region [4–6]. Due to certain inherent advantages, the boundary element method (BEM) can also be used as an alternative numerical scheme for stochastic analysis in conjunction with the perturbation techniques [7,8]. The perturbation stochastic BEM has been successfully applied in solving stochastic elastostatic problems [9–11], stochastic elastodynamic problems [12,13], stochastic wave motion problems [14], vibroacoustic problems [15], stochastic potential problems [16], stochastic heat conduction problems [17], and stochastic groundwater flow problems [18,19], etc. A boundary element-based method is proposed by Spanos [20] for random vibration problems of continuous structures, in which the KL expansion is used to express the random excitation efficiently.

In recent years, a modified approach to the traditional stochastic BEM, i.e. the stochastic SFBEM, has been developed by the authors and applied to stochastic elastostatic problems [21,22], stochastic plate bending problems [23,24], and stochastic fracture problems [25]. SFBEM was originally proposed for the analysis of deterministic problems [26], in which nonsingular integral equations are derived using the fictitious boundary technique, and B-spline functions are adopted to approximate the unknown fictitious loads with the use of the boundary-segment-least-square technique for eliminating the boundary residues. SFBEM has been proved to have high accuracy and efficiency in general.

In this study, the stochastic SFBEM is extended to stochastic modal analysis of plane elastic problems with random fields. Different from the central difference formulation for approximation of the derivatives of the random fields [22,24], the KL expansion in conjunction with the Galerkin techniques [27] is employed in this paper to represent the random fields in terms of a linear combination of the orthogonal basis functions modulated with a set of

* Corresponding author at: School of Civil Engineering and Transportation, South China University of Technology, Guangzhou 510640, PR China.
E-mail address: fanxm@scut.edu.cn (X. Fan).

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uncorrelated random variables, so that the derivatives of the random fields can be derived analytically and the number of random variables needed for description of the random fields can be reduced significantly. The feasibility and effectiveness of the present approach are validated using several numerical examples. A good agreement can be observed with the results of MCS, and a higher efficiency can be obtained as compared with the perturbation stochastic FEM.

2. Stochastic governing differential equations

Without loss of generality, consider the plane stress problem with varied elasticity modulus $E(x,y)$ and mass density $m(x,y)$. The governing differential equations for mode shapes in modal analysis of plane elastic problem can be expressed as [28]

$$\left.\begin{aligned} &\frac{\partial}{\partial x}\left\{\frac{E(x,y)}{1-\mu^2}\left[\frac{\partial U(x,y)}{\partial x}+\mu\frac{\partial V(x,y)}{\partial y}\right]\right\}+\frac{\partial}{\partial y}\left\{\frac{E(x,y)}{2(1+\mu)}\left[\frac{\partial V(x,y)}{\partial x}+\frac{\partial U(x,y)}{\partial y}\right]\right\}+\lambda m(x,y)U(x,y)=0\\ &\frac{\partial}{\partial y}\left\{\frac{E(x,y)}{1-\mu^2}\left[\frac{\partial V(x,y)}{\partial y}+\mu\frac{\partial U(x,y)}{\partial x}\right]\right\}+\frac{\partial}{\partial x}\left\{\frac{E(x,y)}{2(1+\mu)}\left[\frac{\partial V(x,y)}{\partial x}+\frac{\partial U(x,y)}{\partial y}\right]\right\}+\lambda m(x,y)V(x,y)=0 \end{aligned}\right\} \tag{1}$$

where μ is the Poisson's ratio; (x,y) is the Cartesian coordinate of a point in the plane domain considered; $U(x,y)$ and $V(x,y)$ denote displacement modes in horizontal and vertical directions respectively; and λ is the dynamic eigenvalue.

Assume $E(x,y)$ is a homogeneous random field, while $m(x,y)$ is a general random field. They can be expressed as

$$\left.\begin{aligned} &E(x,y)=E_E+\delta E(x,y)\\ &m(x,y)=m_E(x,y)+\delta m(x,y) \end{aligned}\right\} \tag{2}$$

where E_E and $\delta E(x,y)$ are the mean and deviation of $E(x,y)$, respectively; and $m_E(x,y)$ and $\delta m(x,y)$ represent the mean and deviation of $m(x,y)$, respectively. Note that the mean value E_E is a constant, since $E(x,y)$ is assumed to be a homogeneous random field.

Due to the random influence of $E(x,y)$ and $m(x,y)$, the displacement modes and stress modes are also random fields, and the dynamic eigenvalue is a random variable. They can be written as

$$\left.\begin{aligned} &U(x,y)=U_E(x,y)+\delta U(x,y)\\ &V(x,y)=V_E(x,y)+\delta V(x,y)\\ &\sigma_x(x,y)=\sigma_{xE}(x,y)+\delta\sigma_x(x,y)\\ &\sigma_y(x,y)=\sigma_{yE}(x,y)+\delta\sigma_y(x,y)\\ &\tau_{xy}(x,y)=\tau_{xyE}(x,y)+\delta\tau_{xy}(x,y)\\ &\lambda=\lambda_E+\delta\lambda \end{aligned}\right\} \tag{3}$$

where $\sigma_x(x,y)$, $\sigma_y(x,y)$ and $\tau_{xy}(x,y)$ are normal and shear stress modes, respectively; and $(\cdot)_E$ and $\delta(\cdot)$ represent the means and deviations of the random quantities considered.

Substituting Eqs. (2) and (3) into Eq. (1), and neglecting the second and higher order terms of the deviations and their derivatives, one can obtain

$$\left.\begin{aligned} &\frac{\partial}{\partial x}\left\{\frac{E_E}{1-\mu^2}\left[\frac{\partial U_E(x,y)}{\partial x}+\frac{\partial\delta U(x,y)}{\partial x}+\mu\frac{\partial V_E(x,y)}{\partial y}+\mu\frac{\partial\delta V(x,y)}{\partial y}\right]+\frac{\delta E(x,y)}{1-\mu^2}\left[\frac{\partial U_E(x,y)}{\partial x}+\mu\frac{\partial V_E(x,y)}{\partial y}\right]\right\}\\ &+\frac{\partial}{\partial y}\left\{\frac{E_E}{2(1+\mu)}\left[\frac{\partial V_E(x,y)}{\partial x}+\frac{\partial\delta V(x,y)}{\partial x}+\frac{\partial U_E(x,y)}{\partial y}+\frac{\partial\delta U(x,y)}{\partial y}\right]+\frac{\delta E(x,y)}{2(1+\mu)}\left[\frac{\partial V_E(x,y)}{\partial x}+\frac{\partial U_E(x,y)}{\partial y}\right]\right\}\\ &+\lambda_E m_E(x,y)U_E(x,y)+\delta\lambda m_E(x,y)U_E(x,y)+\lambda_E\delta m(x,y)U_E(x,y)+\lambda_E m_E(x,y)\delta U(x,y)=0\\ &\frac{\partial}{\partial y}\left\{\frac{E_E}{1-\mu^2}\left[\frac{\partial V_E(x,y)}{\partial y}+\frac{\partial\delta V(x,y)}{\partial y}+\mu\frac{\partial U_E(x,y)}{\partial x}+\mu\frac{\partial\delta U(x,y)}{\partial x}\right]+\frac{\delta E(x,y)}{1-\mu^2}\left[\frac{\partial V_E(x,y)}{\partial y}+\mu\frac{\partial U_E(x,y)}{\partial x}\right]\right\}\\ &+\frac{\partial}{\partial x}\left\{\frac{E_E}{2(1+\mu)}\left[\frac{\partial V_E(x,y)}{\partial x}+\frac{\partial\delta V(x,y)}{\partial x}+\frac{\partial U_E(x,y)}{\partial y}+\frac{\partial\delta U(x,y)}{\partial y}\right]+\frac{\delta E(x,y)}{2(1+\mu)}\left[\frac{\partial V_E(x,y)}{\partial x}+\frac{\partial U_E(x,y)}{\partial y}\right]\right\}\\ &+\lambda_E m_E(x,y)V_E(x,y)+\delta\lambda m_E(x,y)V_E(x,y)+\lambda_E\delta m(x,y)V_E(x,y)+\lambda_E m_E(x,y)\delta V(x,y)=0 \end{aligned}\right\} \tag{4}$$

Taking the expectances of the terms on both sides of Eq. (4), one has

$$\left.\begin{aligned} &\frac{E_E}{1-\mu^2}\left[\frac{\partial^2 U_E(x,y)}{\partial x^2}+\frac{1-\mu}{2}\frac{\partial^2 U_E(x,y)}{\partial y^2}+\frac{1+\mu}{2}\frac{\partial^2 V_E(x,y)}{\partial x\partial y}\right]+F^1(x,y)=0\\ &\frac{E_E}{1-\mu^2}\left[\frac{\partial^2 V_E(x,y)}{\partial y^2}+\frac{1-\mu}{2}\frac{\partial^2 V_E(x,y)}{\partial x^2}+\frac{1+\mu}{2}\frac{\partial^2 U_E(x,y)}{\partial x\partial y}\right]+F^2(x,y)=0 \end{aligned}\right\} \tag{5}$$

where $F^l(x,y)(l=1,\ 2)$ can be expressed as

$$\left.\begin{aligned} &F^1(x,y)=\lambda_E m_E(x,y)U_E(x,y)\\ &F^2(x,y)=\lambda_E m_E(x,y)V_E(x,y) \end{aligned}\right\} \tag{6}$$

Substitution of Eqs. (5) and (6) into Eq. (4) yields

$$\left.\begin{aligned} &\frac{E_E}{1-\mu^2}\left[\frac{\partial^2\delta U(x,y)}{\partial x^2}+\frac{1-\mu}{2}\frac{\partial^2\delta U(x,y)}{\partial y^2}+\frac{1+\mu}{2}\frac{\partial^2\delta V(x,y)}{\partial x\partial y}\right]+\delta F^1(x,y)=0\\ &\frac{E_E}{1-\mu^2}\left[\frac{\partial^2\delta V(x,y)}{\partial y^2}+\frac{1-\mu}{2}\frac{\partial^2\delta V(x,y)}{\partial x^2}+\frac{1+\mu}{2}\frac{\partial^2\delta U(x,y)}{\partial x\partial y}\right]+\delta F^2(x,y)=0 \end{aligned}\right\} \tag{7}$$

where $\delta F^l(x,y)(l=1,\ 2)$ can be expressed as

$$\left.\begin{aligned} &\delta F^1(x,y)=\frac{\partial}{\partial x}\left\{\frac{\delta E(x,y)}{1-\mu^2}\left[\frac{\partial U_E(x,y)}{\partial x}+\mu\frac{\partial V_E(x,y)}{\partial y}\right]\right\}+\frac{\partial}{\partial y}\left\{\frac{\delta E(x,y)}{2(1+\mu)}\left[\frac{\partial V_E(x,y)}{\partial x}+\frac{\partial U_E(x,y)}{\partial y}\right]\right\}\\ &\quad+[\delta\lambda m_E(x,y)U_E(x,y)+\lambda_E\delta m(x,y)U_E(x,y)+\lambda_E m_E(x,y)\delta U(x,y)]\\ &\delta F^2(x,y)=\frac{\partial}{\partial y}\left\{\frac{\delta E(x,y)}{1-\mu^2}\left[\frac{\partial V_E(x,y)}{\partial y}+\mu\frac{\partial U_E(x,y)}{\partial x}\right]\right\}+\frac{\partial}{\partial x}\left\{\frac{\delta E(x,y)}{2(1+\mu)}\left[\frac{\partial V_E(x,y)}{\partial x}+\frac{\partial U_E(x,y)}{\partial y}\right]\right\}\\ &\quad+[\delta\lambda m_E(x,y)V_E(x,y)+\lambda_E\delta m(x,y)V_E(x,y)+\lambda_E m_E(x,y)\delta V(x,y)] \end{aligned}\right\} \tag{8}$$

Eqs. (5) and (7) are the governing differential equations for the means and deviations of displacement modes, respectively. Obviously, if the terms $F^l(x,y)(l=1,\ 2)$ in Eq. (5) and the terms $\delta F^l(x,y)(l=1,\ 2)$ in Eq. (7) are regarded as equivalent loads, Eqs. (5) and (7) have the same forms as the governing differential equations with respect to displacements in elastostatic problems. Thus, the displacement fundamental solutions corresponding to elastostatic problems can be employed to calculate the means and deviations of the displacement modes in stochastic elastodynamic problems.

3. Stochastic differential relationship between stress and displacement modes

Based on the constitutive law and kinematic relations, the differential relationship between stress and displacement modes for plane stress problems can be expressed as [28]

$$\left.\begin{aligned} &\sigma_x(x,y)=\frac{E(x,y)}{1-\mu^2}\left[\frac{\partial U(x,y)}{\partial x}+\mu\frac{\partial V(x,y)}{\partial y}\right]\\ &\sigma_y(x,y)=\frac{E(x,y)}{1-\mu^2}\left[\frac{\partial V(x,y)}{\partial y}+\mu\frac{\partial U(x,y)}{\partial x}\right]\\ &\tau_{xy}(x,y)=\frac{E(x,y)}{2(1+\mu)}\left[\frac{\partial V(x,y)}{\partial x}+\frac{\partial U(x,y)}{\partial y}\right] \end{aligned}\right\} \tag{9}$$

Substituting Eqs. (2) and (3) into Eq. (9), and neglecting the second and higher order terms of the deviations and their derivatives, the means and deviations of stress modes can be obtained respectively as

$$\left.\begin{aligned} &\sigma_{xE}(x,y)=\frac{E_E}{1-\mu^2}\left[\frac{\partial U_E(x,y)}{\partial x}+\mu\frac{\partial V_E(x,y)}{\partial y}\right]\\ &\sigma_{yE}(x,y)=\frac{E_E}{1-\mu^2}\left[\frac{\partial V_E(x,y)}{\partial y}+\mu\frac{\partial U_E(x,y)}{\partial x}\right]\\ &\tau_{xyE}(x,y)=\frac{E_E}{2(1+\mu)}\left[\frac{\partial V_E(x,y)}{\partial x}+\frac{\partial U_E(x,y)}{\partial y}\right] \end{aligned}\right\} \tag{10}$$