Engineering Analysis with Boundary Elements 66 (2016) 119–128

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

# Isogeometric shape design sensitivity analysis of elasticity problems using boundary integral equations

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ARTICLE INFO

*Article history:*
Received 27 June 2015
Received in revised form
19 November 2015
Accepted 14 January 2016
Available online 2 March 2016

*Keywords:*
Isogeometric method
Boundary integral equation
Shape design sensitivity
NURBS basis function
Exact geometry
Higher-order geometric information

ABSTRACT

Using boundary integral equations and isogeometric approach, a shape design sensitivity analysis (DSA) method is developed for two dimensional elastic structures. In the isogeometric approach, NURBS basis functions in CAD systems are directly utilized in response analysis, which enables a seamless incorporation of exact geometry and higher continuity into computational framework. To enhance the accuracy of shape design sensitivity, the CAD-based higher-order geometric information such as curvature, normal, and tangential vector is exactly embedded in the sensitivity expressions. In boundary integral formulation, shape design velocity field is decomposed into normal and tangential components, which significantly affect the accuracy of shape design sensitivity. Also, the proposed boundary-based method does not require the tedious design parameterization of internal domain. Through the numerical examples, the developed shape DSA method turns out to be more accurate than conventional finite element based one.



## 1. Introduction

Ever since the framework of isogeometric analysis (IGA) method was established by Hughes et al. [1], the IGA method that employs the same basis functions as used in the CAD model has shown many advantages over the standard finite element analysis (FEA). The geometric approximation which is inherent in the finite element mesh could end up with accuracy problems in response analysis and more adversely in design sensitivity analysis. Besides, the isogeometric method has a major feature such as the CAD based parameterization of field variables in an isoparametric manner, which thus requires no further communication with the CAD systems during mesh refinement process. In applying the IGA to shape design optimization problems, accurate design sensitivity analysis (DSA) is essential. Based on the shape DSA theory [2], Cho and Ha [3] showed the applicability and accuracy of the isogeometric shape DSA method for the displacement and stress measures. In addition to the benefits of IGA, the isogeometric DSA has the following advantages: First, it provides more accurate sensitivity of complicated geometries including higher order effects such as curvature, normal, and tangential vector information. The NURBS functions of higher continuity offer a much more compact representation of response and sensitivity of structures than the standard basis functions do, yielding better accuracy even at the same polynomial order. Second, it vastly simplifies the design modification of complicated geometry without communication with the CAD description. Since the NURBS basic functions are used in both the isogeometric response and the sensitivity analyses, design modifications are easily obtainable using the adjustment of control points which represent the geometric model. The design velocity field, defined as the mapping rate between the original and the perturbed domains, plays an important role in computing the shape design sensitivity coefficients. The combination of isoparametric mapping and boundary displacement methods is known to be a natural way to obtain the design velocity field [4]. When using the conventional FEA, the high inter-element continuity of design space is not guaranteed and the geometric information such as curvature, normal, and tangential vectors are not accurate enough. On the other hand, in the isogeometric DSA, the sufficient continuity and the accurate geometric information can be obtained over the whole design space even at coarse mesh so that more accurate shape sensitivity can be expected.

The boundary integral equation (BIE) method for potential problems was developed by Jaswon [5] and Symm [6] as a pioneering work and extended to elasticity problems by Cruse [7]. Since then, the BIE method has extended its applications to heat conduction, acoustic, and crack propagation problems by means of a powerful and alternative numerical method. However, singularity problems arise due to the singular fundamental solution

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http://dx.doi.org/10.1016/j.enganabound.2016.01.010
0955-7997/

expressed as Green functions. The difficulty of dealing with these singularities has been a main issue in the application of BIE method in various engineering problems, which naturally led to several integration schemes to handle the singular integrals. The computation of Cauchy Principle Value (CPV) for strong singular integrals was proposed by Guiggiani and Casalini [8] as a direct approach and a rigid body method was developed by Brebbia [9] as an indirect approach. Liu and Rudolphi [10] shows the integral identities for fundamental solutions without the computation of CPV. Meanwhile, weakly singular integration can be implemented based on the transformation method by Telles [11]. Recently, a BIE method employing the isogeometric approach was developed together with the collocation method to precisely locate the field and the source points [12].

The BIE-based shape optimization method has been developed for several decades. Using the BIE and the adjoint variable method (AVM) in continuum approach, Choi and Kwak derived shape DSA methods for the self-adjoint elliptic boundary value problems [13] and the applications for general stress-constrained problems in terms of tangential and normal design velocity fields [14]. Xin et al. [15] derived shape design sensitivity by direct differentiation method (DDM) for solids undergoing small-strain, small-rotation, elasto-visco-plastic deformation and carried out shape optimization for plate problems. Yamazaki et al. [16] derived stress sensitivity based on the DDM of discrete boundary integral equations and determined the optimal shape of minimum weight subjected to stress constraints in three dimensional problems. The BIE-based DSA method was further applied to the shape optimization for many engineering problems such as heat conduction [17], acoustic problems [18], and so on. Also, an extension to isogeometric shape optimization was performed for elastic problems [19]. However, the shape sensitivity equation was derived in discrete form and did not include any discussions and verification of the derived shape sensitivity in that literature.

This paper is organized as follows; in Section 2, we describe the construction of NURBS basis functions, which may have up to $(p-1)$ continuous derivatives across element boundaries where $p$ is the order of underlying polynomial. We explain the isogeometric BIE method based on the NURBS. In Section 3, we derive the isogeometric BIE shape design sensitivity, where the geometric effects seem to have profound effects on the shape design sensitivity. In Section 4, demonstrative numerical examples are presented to verify the accuracy of isogeometric sensitivity by comparing with the exact solution of partial differential equation or the conventional BIEM solution. Finally, we draw conclusions, which present the superior points of proposed isogeometric shape DSA method.

## 2. Isogeometric boundary integral equation

### 2.1. NURBS basis function

In the IGA, the solution space is represented in terms of the same basis functions as used in describing the geometry. The IGA has several advantages over the conventional FEA: *geometric exactness* and *simple refinements* due to the use of NURBS basis functions which are based on B-splines. Consider a set of knots $\boldsymbol{\Xi}$ in one-dimensional parametric space.

$$\boldsymbol{\Xi}=\left\{\xi_1,\xi_2,\cdots,\xi_{n+p+1}\right\}, \tag{1}$$

where $p$ and $n$ are the order of basis function and the number of control points, respectively. The B-spline basis functions are defined, recursively, as

$$N_i^0(\xi)=\begin{cases}1 & \text{if } \xi_i\le\xi<\xi_{i+1}\\ 0 & \text{otherwise}\end{cases},\quad (p=0) \tag{2}$$

and

$$N_i^p(\xi)=\frac{\xi-\xi_i}{\xi_{i+p}-\xi_i}N_i^{p-1}(\xi)+\frac{\xi_{i+p+1}-\xi}{\xi_{i+p+1}-\xi_{i+1}}N_{i+1}^{p-1}(\xi),\quad (p=1,2,3,\ldots). \tag{3}$$

Using the B-spline basis function $N_i^p(\xi)$ and the corresponding weight $w_i$, a NURBS basis function $R_i^p(\xi)$ is defined as

$$R_i^p(\xi)=\frac{N_i^p(\xi)w_i}{\sum_{j=1}^{n}N_j^p(\xi)w_j}. \tag{4}$$

Generally, the isogeometric approach using higher order basis functions offers higher regularity than the conventional FEA does. For a given $n$ pairs of $p$-$th$ order NURBS basis function $R_i^p(\xi)$ and the corresponding control point $\mathbf{B}_i$, a NURBS curve $\mathbf{C}$ is obtained by

$$\mathbf{C}(\xi)=\sum_{i=1}^{n}R_i^p(\xi)\mathbf{B}_i. \tag{5}$$

For the details of NURBS geometry, interested readers may consult Rogers [20], Piegl and Tiller [21]. The constructed NURBS basis functions possess the property of affine covariance and $(p-1)$ continuous differentiability. If the knots are repeated $k$-times, the continuity of NURBS basis functions decreases by $k$ as well.

### 2.2. Boundary integral equation for plane elasticity

Consider an open domain $\Omega$ bounded by a closed surface $\Gamma$ which is sufficiently smooth and composed of two disjointed parts as $\Gamma=\overline{\Gamma_D\cup\Gamma_N}$, where $\Gamma_D$ and $\Gamma_N$ are the displacement and traction boundaries, respectively. $\mathbf{n}$ is an outward unit vector that is normal to the boundary $\Gamma$ and $\mathbf{b}$ is a body force intensity (Fig. 1).

For a generic point $\mathbf{x}$ in the domain, the governing equation for plane elasticity is given by

$$\sigma_{ij,j}+b_i=0,\quad \mathbf{x}\in\Omega, \tag{6}$$

with the following boundary conditions

$$u_i=\overline{u}_i,\quad \mathbf{x}\in\Gamma_D \tag{7}$$

and

$$t_i=\sigma_{ij}n_j=\overline{t}_i,\quad \mathbf{x}\in\Gamma_N, \tag{8}$$

where $\overline{u}_i$ and $\overline{t}_i$ are the prescribed displacement and traction, respectively. For a unit concentrated load at a source point $\hat{\mathbf{x}}$, the arbitrary function $v_i$ should satisfy the following.

$$\sigma_{ij,j}(\mathbf{v})=-\delta(\mathbf{x}-\hat{\mathbf{x}})e_i, \tag{9}$$

$$v_i=U_{ij}e_j, \tag{10}$$

and

$$\sigma_{ij}(\mathbf{v})n_j=T_{ij}e_j, \tag{11}$$

where $e_i$ is a unit vector. The fundamental solutions of two-dimensional problem for a field point $\mathbf{x}$ are given by

$$U_{ij}(\mathbf{x},\hat{\mathbf{x}})=\frac{1}{8\pi\mu(1-\nu)}\left[(3-4\nu)\ln\frac{1}{r}\delta_{ij}+r_{,i}r_{,j}\right] \tag{12}$$

and

$$T_{ij}(\mathbf{x},\hat{\mathbf{x}})=-\frac{1}{4\pi(1-\nu)r}\left[\frac{\partial r}{\partial n}\left[(1-2\nu)\delta_{ij}+2r_{,i}r_{,j}\right]+(1-2\nu)\left(n_ir_{,j}-n_jr_{,i}\right)\right], \tag{13}$$