Engineering Analysis with Boundary Elements 67 (2016) 1–12

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

A fast multipole method accelerated adaptive background cell-based domain integration method for evaluation of domain integrals in 3D boundary element method

Wei Zhou, Qiao Wang*, Yonggang Cheng, Gang Ma

State Key Laboratory of Water Resources and Hydropower Engineering Science, Wuhan University, Wuhan 430072, China

ARTICLE INFO

Article history:
Received 18 June 2015
Received in revised form
6 December 2015
Accepted 2 March 2016
Available online 16 March 2016

Keywords:
Cell-based domain integration method
Domain integrals
Boundary element method
Fast multipole method

ABSTRACT

A background cell-based domain integration method is proposed in this paper for evaluating domain integrals in 3D problems. The cells are created by an adaptive oct-tree structure based on the information of boundary elements, and no other discretization is needed. Cells that contain boundary elements can be subdivided into smaller sub-cells adaptively to obtain the desired accuracy according to the sizes and levels of the boundary elements. Applying the method directly in the boundary element method is time-consuming since the time complexity is $O(NM)$, where N and M are the numbers of nodes and cells, respectively. The fast multipole method is coupled with the cell-based domain integration method to further accelerate the computational efficiency, and the main formulations are introduced in this paper. Numerical examples have demonstrated the accuracy and efficiency of the proposed method.

1. Introduction

The boundary element method (BEM) [1,2] has attracted the attention of many researchers because only the boundary of the domain needs to be discretized. However, for non-homogenous problems, the boundary integral equation often includes domain integrals, which have restricted the application of BEM. This problem also exists in some other boundary type methods, such as boundary node method (BNM) [3], Galerkin boundary node method (GBNM) [4], hybrid boundary node method (Hybrid BNM) [5–7] and boundary face method (BFM) [8].

A direct domain integration method [9] is the most natural way to evaluate domain integrals. However, the domain needs to be partitioned into volume elements by applying this method, and the BEM or other boundary type method will no longer be a boundary-only discretization method. To maintain the characteristic that only the boundary needs to be discretized, many researchers focus on translating the domain integrals into boundary integrals. Several methods have been proposed so far, and a review can be found in [10]. In the dual reciprocity method (DRM) [11], the non-homogeneous terms are approximated with radial basis functions (RBF), and the domain integrals can be translated to the boundary by a second reciprocity. The multiple reciprocity method (MRM) [12] and the particular solution method (PSM) [13] are similar to DRM. MRM can be considered as an extension of the DRM, and the reciprocity theorem is repeatedly applied using a sequence of higher order fundamental solutions to transform the domain integrals into the boundary. In PSM, an approximate particular solution is constructed rather than performing a second reciprocity. Recently, Gao [14,15] proposed a new approach for evaluating two and three dimensional domain integrals with boundary-only discretization called the radial integration method (RIM). In this approach, the domain integral is transformed into an equivalent boundary integral using straight-path integrals emanating from the source point. Similar work can be found in [16–18]. Instead of attempting to transform the domain integrals into boundary integrals, Koehler et. al [19] used the "Galerkin vector" to represent the Green's function, and the volume integral was decomposed into a boundary integral together with a volume integral, wherein the source function was everywhere zero on the boundary.

Although the BEM can reduce one dimension of the problem, it has a dense system matrix that needs at least $O(N^2)$ operations to be solved, where N is the total degrees of freedom (DOFs). This problem has been noticed by many researchers, and some fast algorithms have been applied to accelerate BEM, such as the fast multipole method (FMM) [20,21] and the adaptive cross approximation [22]. Unfortunately, the computation for domain integrals in BEM is also time-consuming, even though the domain integrals can be converted to boundary integrals. The main reason lies in the time complexities of the applied methods, and the estimations of the time complexities for some methods that are applied to

* Corresponding author.
E-mail address: qiaowang@whu.edu.cn (Q. Wang).

http://dx.doi.org/10.1016/j.enganabound.2016.03.002
0955-7997/

evaluate domain integrals are proposed in the second section of this paper.

In the research of Ingber et al. [23], the classical direct volume integration method was reported to be a significantly better choice for computing domain integrals compared to DRM and PSM. In their work, FMM was used to accelerate the integration in the 2D case. A recent work that used FMM to fast evaluate the Newton potentials in the Galerkin BEM with volume elements can be found in [24], and related work using the fast Fourier transform (FFT) accelerated techniques can be found in [25,26].

However, direct volume integration methods often need to discretize the domain into elements, which is difficult for complex domains and loses the boundary-only nature of the BEM. This paper presents an adaptive background cell-based domain integration method (CDIM) to overcome this disadvantage, and the method is accelerated by FMM in 3D problems. The adaptive background cell-based domain integration method uses the direct domain integration method and no further translation is applied. The cells are constructed from the boundary elements and an adaptive tree structure. No volume elements are needed in the proposed method, and the background cells that contain the boundary elements can be adaptively subdivided into smaller subcells to obtain sufficient accuracy. The cells are obtained from the information of the boundary elements without other discretization in the proposed method. The time complexity of this procedure is $O(NM)$ when applied directly, and significant time is required for complex or large-scale problems, where N and M are the numbers of nodes and cells, respectively. To reduce the computational time, a fast algorithm can be used. FMM [20,21] has been widely used in BEM [27–30] and some boundary type meshless methods [31,32] to accelerate the matrix-vector multiplication in the iterative solver. In the background cell-based domain integration method, FMM can be used directly to accelerate the domain integration and the background cells can be used as the adaptive tree structure in FMM directly. Similarly, the FMM accelerated method [24] often needs to partition the domain into volume elements, and FFT accelerated techniques [25,26] used uniform grids to approximate the domain. However, the method proposed in this paper is a boundary-only discretization method without volume elements like the FFT accelerated techniques, but adaptive background cells that are created from the boundary rather than uniform grids elements are applied.

This paper is organized as follows. The complexities of some well-known methods for computing domain integrals are analyzed in the second section. The third section introduces the adaptive background-cell based domain integration method for BEM, and followed by the acceleration by FMM in the fourth section. Finally, numerical examples are studied to demonstrate the efficiency and accuracy of the proposed method.

2. Time complexities of methods for computing domain integrals

In this section, the time complexities of some methods for computing domain integrals are estimated to demonstrate the necessity of applying fast algorithms to accelerate the integrals. The boundary element method for the Poisson equation will be reviewed to describe the domain integrals.

Consider a region Ω bounded by boundary Γ, the direct formulation of the boundary integral equation for Poisson equation can be written as

$$c(\mathbf{x})u(\mathbf{x})+\int_\Gamma q^*(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) = \int_\Gamma u^*(\mathbf{x},\mathbf{y})q(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y})-\int_\Omega u^*(\mathbf{x},\mathbf{y})b(\mathbf{y})\mathrm{d}\Omega(\mathbf{y}) \quad (1)$$

where $\mathbf{x}$ and $\mathbf{y}$ are points on the boundary, and $c(\mathbf{x})$ is a coefficient related to the geometry of the boundary at $\mathbf{x}$. u and q are the fields of the potential and normal flux on the boundary, and b is the nonhomogeneous term in the Poisson equation. $u^*(\mathbf{x},\mathbf{y})$ is the fundamental solution of the Laplace equation, which can be written in the 3D case as

$$u^*(\mathbf{x},\mathbf{y})=\frac{1}{4\pi r(\mathbf{x},\mathbf{y})} \quad (2)$$

and

$$q^*(\mathbf{x},\mathbf{y})=\frac{\partial u^*}{\partial \mathbf{n}} \quad (3)$$

where $r(\mathbf{x},\mathbf{y})$ is the distance between $\mathbf{x}$ and $\mathbf{y}$, and $\mathbf{n}$ is the unit outward normal at the boundary point.

The domain integral in Eq. (1) can be defined as

$$I(\mathbf{x})=\int_\Omega u^*(\mathbf{x},\mathbf{y})b(\mathbf{y})\mathrm{d}\Omega(\mathbf{y}) \quad (4)$$

2.1. Direct domain integration method

The integral in Eq. (4) can be computed directly by volume elements or cells as

$$I(\mathbf{x})=\sum_{i=1}^{M}\int_{\Omega_i} u^*(\mathbf{x},\mathbf{y})b(\mathbf{y})\mathrm{d}\Omega_i(\mathbf{y}) \quad (5)$$

where M is the number of volume elements. The time complexity is $O(NM)$ if Eq. (5) is used to compute the domain integrals in BEM, where N is the number of evaluation nodes $\mathbf{x}$. This method is not suitable for large-scale computation, and it should be accelerated by some fast algorithms.

2.2. Dual reciprocity method

In DRM, the function b can be approximated by a set of interpolation functions at a certain number of boundary nodes and internal nodes. The interpolating scheme is given as

$$b\approx\sum_{j=1}^{N+L}\alpha_j f_j \quad (6)$$

where N and L are the numbers of boundary nodes and nodes in the domain, and f_j and α_j are the interpolating function and the corresponding coefficient. The main idea in DRM is to express f_j as the Laplacian of another function ψ_j such that

$$\nabla^2\psi_j=f_j \quad (7)$$

By applying the reciprocity theorem, the domain integral in Eq. (4) can be converted to the boundary as

$$I(\mathbf{x})=\sum_{j=1}^{N+L}\alpha_j\left\{c(\mathbf{x})\psi_j(\mathbf{x})+\int_\Gamma \psi_j(\mathbf{y})q^*(\mathbf{x},\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) -\int_\Gamma \frac{\partial\psi_j(\mathbf{y})}{\partial\mathbf{n}}u^*(\mathbf{x},\mathbf{y})\mathrm{d}\Gamma(\mathbf{y})\right\} \quad (8)$$

Eq. (8) can be computed by discretizing the boundary into elements, and the time complexity is $O(nNM)$, where $n=N+L$ and M is the number of boundary elements. α_j can be computed by solving Eq. (6), and the time complexity is $O(n^2)$ for an iterative solver.