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Improved localized radial basis function collocation method for multi-dimensional convection-dominated problems

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ABSTRACT

In this paper, the localized radial basis function collocation method (LRBFCM) is combined with the partial upwind scheme for solving convection-dominated fluid flow problems. The localization technique adopted in LRBFCM has shown to be effective in avoiding the well known ill-conditioning problem of traditional meshless collocation method with globally defined radial basis functions (RBFs). For convection–diffusion problems with dominated convection, stiffness in the form of boundary/interior layers and shock waves emerge as convection overwhelms diffusion. We show in this paper that these kinds of stiff problems can be well tackled by combining the LRBFCM with partial upwind scheme. For verification, several numerical examples are given to demonstrate that this scheme improves the LRBFCM in providing stable, accurate, and oscillation-free solutions to one- and two-dimensional Burgers' equations with shock waves and singular perturbation problems with turning points and boundary layers.

1. Introduction

In modelling physical convection–diffusion phenomena of heat diffusion and particle/fluid flows, the following two typical form of partial differential equations (PDEs) are commonly used:

$$\frac{\partial u}{\partial t} - \alpha_1 \Delta u + \boldsymbol{\beta}_1 \cdot \nabla u = f_1, \tag{1}$$

with given initial and boundary conditions for evolutionary convection–diffusion problems; and

$$-\alpha_2 \Delta u + \boldsymbol{\beta}_2 \cdot \nabla u = f_2, \tag{2}$$

with given boundary condition for steady convection–diffusion problems. Here, $\alpha_1, \alpha_2, \boldsymbol{\beta}_1 = (\beta_1^{(1)}, \beta_2^{(1)}, \ldots, \beta_N^{(1)})$ and $\boldsymbol{\beta}_2 = (\beta_1^{(2)}, \beta_2^{(2)}, \ldots, \beta_N^{(2)})$ denote the diffusion and convection coefficients and f_1, f_2 are source terms.

With the rapid advancement of computer technology, numerical methods for simulating real physical phenomena through solving PDEs are vastly developed and analyzed. Among these the traditional finite difference method (FDM), finite element method (FEM), and finite volume method (FVM) are most commonly utilized. Due to the requirement of generating grids/meshes in these methods, these traditional methods are difficult to solve high-dimensional and complex geometrical problems. The development of meshless methods which require only arbitrarily distributed set of nodes has drawn the attention of many researchers [1,2]. Among these meshless methods, the radial basis function collocation method (RBFCM) has been vastly applied to solve various kinds of physical problems [3–6] with theoretical proofs on solvability and convergence presented in [7–9]. Since most of the radial basis functions used in RBFCM are continuously differentiable smooth functions, the method is known as Global Radial Basis Functions Collocation Method (GRBFCM). In GRBFCM, the resultant matrix is constructed by taking direct collocation of nodes in the domain and on the boundary. This makes the GRBFCM one of the simplest numerical methods but then the solution is obtained from solving a dense, and hence ill-conditioned, collocation matrix. This limits the applicability of the GRBFCM to solve large scale problems. In the past, various techniques such as Wendland's Compactly Supported RBF [10], preconditioning technique by Ling and Kansa [11], adaptive greedy algorithm by Ling and Schaback [12] and domain decomposition method by Li and Hon [13] have been proposed to tackle this ill-conditioning problem.

More recently, the GRBFCM has been localized to get Local Radial Basis Function Collocation Method (LRBFCM) [14,15], whose main idea is to apply collocation separately on each overlapping sub-domain of the whole domain. This localization dramatically reduces the denseness of the resultant collocation matrix but at the expense of solving many small sub-matrices [14]. The LRBFCM

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not only keeps the meshless and superior accuracy advantages of GRBFCM, but also is less sensitive to the choice of shape parameter and perturbation values [16]. The optimal choice of shape parameter for multiquadric and Gaussian type RBFs has long been an open problem in the development of RBFCM. This allows the LRBFCM the capability to solve many complex physical problems such as phase-change, casting, Navier–Stokes equations and turbulent flow [17,18] with complexly shaped domains [19].

For convection–diffusion equations, it is well known that if the magnitude of the convection coefficient is much greater than the diffusion coefficient, i.e., $|\boldsymbol{\beta}_1| \gg \alpha_1$ in (1) and $|\boldsymbol{\beta}_2| \gg \alpha_2$ in (2), some kinds of stiffness in the form of boundary/interior layers and shock waves emerge. The direct use of FDM and FEM to these convection-dominated fluid flow problems will result in some kinds of instability and oscillatory. In [20,21], the method of one-sided or upwind scheme was adopted in FDM to obtain an oscillation-free numerical solution and later Kopteva [22] extended upwind scheme to solve two-dimensional problems. Since this fully upwind scheme introduces additional artificial diffusion in contrast to the central difference scheme, Pandian [23] proposed a partial upwind scheme by taking a weighted combination of upwind scheme with central difference scheme. Moreover, the upwind technique has also been used in [24–26] with FEM to obtain stable numerical solutions without non-physical oscillations. Furthermore, Shu et al. have applied successfully the Essentially Non-Oscillatory (ENO) and Weighted Essentially Non-Oscillatory (WENO) schemes to obtain non-oscillatory solutions [27–29], in which a nonlinear adaptive procedure was used to choose local stencil automatically for higher accuracy.

For meshless methods, Gu and Liu [30] proposed an adaptive upwind scheme to solve steady convection–diffusion problems. This adaptive upwind scheme was later adopted in RBFCM to solve steady convection–diffusion problems by Chandhini and Sanyasiraju [31]. The comparison given in [32] indicates that the adaptive upwind technique works better than the fully upwind scheme. This adaptive technique was also applied to solve a two-dimensional coupled Burgers' equations by Siraj-ul-Islam et al. [33]. Zahab et al. [34] used a localized collocation meshless method with upwind scheme to model laminar incompressible flows. Shu et al. [35] later applied the local RBF differential quadrature method with upwind scheme to study inviscid compressible flows with shock waves. From the numerical results given in [36], it was found that the upwind scheme does not work well at nodes near the boundary layer when the perturbation parameter is not small. Recently, these upwind techniques have been applied to solve convection-dominated problems [37,38]. For solving hyperbolic problems, the upwind technique is also applicable [39] to get numerical solutions without non-physical oscillations. Besides, Fornberg and Lehto [40] used hyper-viscosity with RBF-generated finite difference (RBF-FD) method to solve convective PDEs. Flyer and Lehto in [41] developed a local refinement algorithm for problems need higher resolution. It was claimed that the effectiveness of RBF-FD will be greatly improve by this local refine method.

To achieve higher accuracy and maintain the advantages of meshless, stable, and non-oscillatory, we combine in this paper the LRBFCM with partial upwind technique to solve time-dependent and steady convection-dominated problems. Numerical results indicate that this improved LRBFCM gives stable and higher accurate numerical solutions without non-physical oscillation. Compared with fully upwind scheme, the adoption of partial upwind scheme reduces the effect of additional diffusion that is brought by upwind scheme. Furthermore, this LRBFCM with partial upwind scheme can approximate the solution with high accuracy even when the convection effect is not so overwhelming in contrast with diffusion. This makes the proposed method better equipped to solve more general stiff problems.

The organization of this paper is as follows: In Section 2, we introduce the LRBFCM and give explicit and implicit schemes to solve boundary value problems (BVPs) and initial boundary value problems (IBVPs). The LRBFCM with partial upwind scheme is then presented for solving problems in both one- and two-dimensional space. Section 3 is then devoted to the numerical verification on the effectiveness and efficiency of LRBFCM with upwind and partial wind schemes for solving the typical one- and two-dimensional Burgers' equations with shock waves and singular perturbed problems with boundary layers. In Section 4, our proposed numerical method is applied to solve convection-dominated problems with variable upwind directions. Conclusions are given in Section 5.

2. LRBFCM for solving PDEs

Suppose $u(\mathbf{x})$ is a function defined in Ω, where $\Omega \subseteq \mathbb{R}^N$ and N is a positive integer. Given a set of pairwise distinct nodes $\Xi \triangleq \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ in $\overline{\Omega} = \Omega \cup \partial\Omega$, the idea of approximating $u(\mathbf{x})$ by global RBFs is to assume a linear combination of RBFs for an approximation of the solution as follows:

$$\tilde{u}(\mathbf{x}) = \sum_{j=1}^{n} \lambda_j \phi(\|\mathbf{x}-\mathbf{x}_j\|) = \Phi(\mathbf{x})\Lambda, \tag{3}$$

where each $\Phi(\mathbf{x}) = [\phi(\|\mathbf{x}-\mathbf{x}_1\|), \phi(\|\mathbf{x}-\mathbf{x}_2\|), \dots, \phi(\|\mathbf{x}-\mathbf{x}_n\|)]$, $\phi(\|\mathbf{x}-\mathbf{x}_j\|)$ is a RBF centered at $\mathbf{x}_j$, $\|\cdot\|$ is the usual Euclidean norm between nodes $\mathbf{x}$ and $\mathbf{x}_j$, and $\Lambda = [\lambda_1, \lambda_2, \dots, \lambda_n]^T$ is a vector of unknowns to be determined.

There are many choices of RBFs in multivariate interpolation [10] among which the Multiquadric (MQ) is one of the most commonly adopted for superior convergence. In GRBFCM, the following linear system is derived from direct collocation of $\tilde{u}(\mathbf{x}_k) = u(\mathbf{x}_k) = u_k$

$$\boldsymbol{\Psi}\Lambda = \mathbf{u},$$

where $u(\mathbf{x}_k), u_k$ denote the exact and approximate solutions, respectively, $\mathbf{u} = [u_1, u_2, \dots, u_n]^T$, and the coefficient matrix $\boldsymbol{\Psi} = (\phi(\|\mathbf{x}_i - \mathbf{x}_j\|))_{1 \le i,j \le n}$ is given as

$$\boldsymbol{\Psi} = \begin{pmatrix} \phi(\|\mathbf{x}_1-\mathbf{x}_1\|) & \phi(\|\mathbf{x}_1-\mathbf{x}_2\|) & \cdots & \phi(\|\mathbf{x}_1-\mathbf{x}_n\|) \\ \phi(\|\mathbf{x}_2-\mathbf{x}_1\|) & \phi(\|\mathbf{x}_2-\mathbf{x}_2\|) & \cdots & \phi(\|\mathbf{x}_2-\mathbf{x}_n\|) \\ \phi(\|\mathbf{x}_3-\mathbf{x}_1\|) & \phi(\|\mathbf{x}_3-\mathbf{x}_2\|) & \cdots & \phi(\|\mathbf{x}_3-\mathbf{x}_n\|) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(\|\mathbf{x}_n-\mathbf{x}_1\|) & \phi(\|\mathbf{x}_n-\mathbf{x}_2\|) & \cdots & \phi(\|\mathbf{x}_n-\mathbf{x}_n\|) \end{pmatrix}_{n\times n}.$$

The invertibility of the above coefficient matrix resulted from multivariate interpolation has been well studied in [42] and hence we have

$$\Lambda = \boldsymbol{\Psi}^{-1}\mathbf{u}. \tag{4}$$

It follows from (3) and (4) that

$$u(\mathbf{x}) \approx \tilde{u}(\mathbf{x}) = \Phi(\mathbf{x})\lambda = \Phi(\mathbf{x})\boldsymbol{\Psi}^{-1}\mathbf{u}.$$

Due to the smoothness of RBFs, derivatives of $u(\mathbf{x})$ can be approximated by differentiating $\tilde{u}(x)$:

$$D^{\boldsymbol{\alpha}} u(\mathbf{x}) \approx D^{\boldsymbol{\alpha}} \tilde{u}(\mathbf{x}) = \sum_{j=1}^{n} \lambda_j D^{\boldsymbol{\alpha}} \phi(\|\mathbf{x}-\mathbf{x}_j\|) = D^{\boldsymbol{\alpha}} \Phi(\mathbf{x})\Lambda,$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_N) \in \mathbb{N}_0^N$, $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$, $D^{\boldsymbol{\alpha}} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \partial x_1^{\alpha_2} \dots \partial x_1^{\alpha_N}}$, $D^{\boldsymbol{\alpha}}\Phi(\mathbf{x}) = [D^{\boldsymbol{\alpha}}\phi(\|\mathbf{x}-\mathbf{x}_1\|), D^{\boldsymbol{\alpha}}\phi(\|\mathbf{x}-\mathbf{x}_2\|), \dots, D^{\boldsymbol{\alpha}}\phi(\|\mathbf{x}-\mathbf{x}_n\|)]$.