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Numerical solution of ideal MHD equilibrium via radial basis functions collocation and moving least squares approximation methods

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ABSTRACT

In this study, two different meshfree methods consisting of the Radial Basis Functions (RBFs) and the Moving Least Square Method (MLS) are applied to solve the Grad–Shafranov (GS) equation for the axisymmetric equilibrium of plasma in the tokamak. The validity and the effectiveness of the proposed schemes are studied by several test problems through absolute and Root Mean Squared (RMS) errors. Although, during the past few years, a meshfree method is normally applied in magnetohydrodynamic (MHD) studies to the numerical solution of partial differential equations (PDEs) but to the best of our knowledge, its application in MHD equilibrium of the tokamak plasma investigations is rare. The future more extensive studies regarding this numerical method would definitely have a significant impact on improving tokamak numerical tools.

1. Introduction

A tokamak is a type of magnetic confinement device, for producing controlled thermonuclear fusion power. In order to achieve controlled magnetic fusion, the development of mathematical models and computational codes to simulate tokamak plasma behavior, is crucial. The acceptable models of plasma in the tokamak ensure successful device design and plasma performance prediction.

Magnetohydrodynamic (MHD) equilibrium of axisymmetric plasma is governed by the Grad–Shafranov (GS) equation, which was derived by Grad and Rubin [1], Shafranov [2] and Lust et al. [3] independently. This equation is a nonlinear, elliptic partial differential equation obtained from the reduction of the ideal MHD equations to two dimensions, often for the case of toroidal axisymmetry. Solving the equilibrium equation with fixed plasma boundary is called fixed boundary, while unfixed plasma boundary is called free boundary problems.

Equilibrium equation is the fundamental of exploring plasma phenomena in a tokamak, including stability, transport and turbulence, and plays an important role in the design of this device. Therefore, giving an accurate and efficient solution would have an important effect on the tokamak fusion research.

To date, various numerical methods in solving tokamak equilibrium equation have been written, validated and incorporated into comprehensive systems for tokamak design and analysis. A number of main numerical methods applied in equilibrium solver codes are the finite difference method (FDM), finite element method (FEM), boundary element method (BEM), Green's function methods and etc.

The FDM has been used in many equilibrium equation solver tools and it is considered to be a powerful numerical method which has been utilized extensively for years. A numerical equilibrium calculation for a tokamak plasma was first carried out by Callen and Dory [4]. They used FDM and Successive Over Relaxation (SOR) algorithm to solve a fixed boundary equilibrium of a tokamak with a circular cross section in cylindrical coordination. Suzuki [5] used FDM and Alternating Direction Implicit (ADI) algorithm combined with the Marder–Weitzner iteration scheme [6] to solve GS equation with free boundary condition. The conductive shells of arbitrary cross sections and external coils are considered in his study. The FDM again was used by Cenacchi et al. [7] to solve this equation with external coils and without a conducting shell. The other efficient solver based on Cyclic Reduction (CR) algorithm was presented by Helton and Wang [8] to study the shaping and control of equilibrium in tokamak with external coils. Johnson et al. [9] also used the same algorithm to solve GS equation with both fixed and free boundary. A few years later, Ling and Jardin [10] applied FDM to prepare a fast equilibrium solver for both fixed and free boundary conditions. In their calculations, the free-boundary formulation was based on the minimization of a mean-square error, while the fixed-boundary solution was based on a variational principle and spectral representation of the coordinates. Moreover, the FDM and its applications in plasma

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physics, especially the GS equation have been investigated in detail by Jardin [11] in his book.

The FEM is also used widely in the tokamak equilibrium solver tools. Lao et al. [12] applied the variational method to find approximate solutions of the GS equation. Semenzato et al. [13], Kerner and Jandel [14] and Gruber et al. [15] solved equilibrium equation by FEM with consideration of flow in their calculations. A numerical code using Hermite bicubic FEM has been developed by Lutjens et al. [16] for the computation of axisymmetric MHD equilibria. A direct variational method based on energy principle was applied by Ludwig [17]. This method uses a spectral representation of the magnetic flux surfaces in terms of Chebyshev polynomials. Blum et al. [18,19] studied tokamak plasma equilibrium and reconstructed some parameters using FEM.

Another useful method is the BEM. This method is particularly suitable since it requires discretization only on the boundary. Itagaki and Fukunaga [20] used this numerical method to solve GS equation for fixed boundary calculations. In their research, the singularity of the fundamental solution, which consists of two elliptic integrals, and the properties of singular integrals was minutely investigated. More study also was carried out by Itaki and Shimoda [21]. In addition, Aydin et al. [22] studied the numerical solution of the GS equation by using the boundary element, the finite element and the differential quadrature methods for rectangular and D-shape plasmas.

In addition to FDM, FEM and BEM, some other methods, such as Green's function, expansion and so on have also been used in equilibrium solver tools [23–26]. Takedai and Tokuda [27] reviewed the computation methods of the MHD equilibrium of a tokamak plasma comprehensively as possible.

Over the last decade, the meshfree method has become an active research area to solve the PDEs numerical solution, particularly in MHD field. Despite the significant improvements of this scheme in different fields, it is rarely applied in tokamak plasma equilibrium calculations.

Imazawa et al. [28] focused on the meshfree methods, especially RBF-MFS (Method of Fundamental Solutions) and KANSA to solve the GS fixed boundary problem. Nath et al. [29,30] also used MFS to solve the same problem. Further studies using this method do not exist in this field any more.

Due to the rarity of tokamak equilibrium studies using the meshfree method, it is expected that more studies using this numerical solution would have a considerable impact on numerical tools in tokamak equilibrium research.

In this paper, the RBFs and MLS methods are applied to calculate the fixed boundary Grad–Shafranov (GS) equation for the axisymmetric equilibrium plasma. Four test problems related to the tokamak plasma are studied and some errors are reported. The obtained results confirm the validity and effectiveness of the proposed schemes.

The organization of this paper is as follows: in Section 2, derivation and properties of the GS equation are summarized. Description of RBFs and MLS methods and their applications to solve the GS equation is presented in Section 3. Section 4 contains the results for some test problems. Finally, this study is concluded in Section 5.

2. The equilibrium equation

In this section the Grad–Shafranov (GS) equation that presents axisymmetric equilibrium configuration of a tokamak plasma is posed.

The equation of motion for a charged-neutral plasma placed in a magnetic field is described by adding Lorentz force to the Navier Stokes equations.

$$\rho\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right)\boldsymbol{v}=\boldsymbol{J}\times\mathbf{B}-\nabla p, \tag{1}$$

where ρ is the mass density, $\boldsymbol{v}$ is the mass flow velocity, p is the plasma pressure, $\mathbf{J}$ is the current density and $\mathbf{B}$ is a magnetic field.

In equilibrium, no time variation is involved ($\partial/\partial t=0$) in calculations. Also, since ion Mach number (the ratio of the plasma velocity to the ion thermal velocity) is much smaller than unit, for almost all situations of fusion interest, neglecting the inertial term is justified and static equilibria can be considered ($\boldsymbol{v}=0$) [31]. Several studies based on meshfree (except for tokamak equilibrium studies) have been carried out without this assumption [32,33]. Finally, the equation to be solved for tokamak plasma equilibrium is simplified to

$$\nabla p=\boldsymbol{J}\times\mathbf{B}. \tag{2}$$

This basic relationship expresses that at the time of equilibrium the plasma pressure and the magnetic forces are in balance. Eq. (2) along with Maxwell's equations

$$\begin{aligned}&\nabla\times\mathbf{B}=\mu_0\mathbf{J},\\&\nabla\cdot\mathbf{B}=0,\end{aligned} \tag{3}$$

describes ideal magnetohydrodynamic equilibria. The μ_0 parameter is the magnetic permeability. Combining this equations and using the scalar functions, $\psi=\psi(R,Z)$ and $f=f(R,Z)$, the current density and magnetic field can be represented as

$$\begin{aligned}&\mathbf{J}=-\Delta^{*}\psi\nabla\phi+\nabla f\times\nabla\phi,\\&\mathbf{B}=\nabla\psi\times\nabla\phi+f\nabla\phi,\end{aligned} \tag{4}$$

where ψ is the poloidal flux function and f is the poloidal current function. Here elliptic operator, Δ^*, in cylindrical coordinates, (R,ϕ,Z), is defined by

$$\Delta^{*}\psi=R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)+\frac{\partial}{\partial Z}\left(\frac{\partial\psi}{\partial Z}\right). \tag{5}$$

The $\partial/\partial\phi=0$ due to assumed axisymmetric. Finally, the poloidal part of the force balance equation reduces to

$$\Delta^{*}\psi=\mu_0RJ_{tr}, \tag{6}$$

where J_{tr} is the toroidal current density. Considering the relations $f=f(\psi)$ and $p=p(\psi)$

$$\mu_0RJ_{tr}=-\left(\mu_0R^2\frac{dp}{d\psi}+f\frac{df}{d\psi}\right). \tag{7}$$

Eq. (6) is a nonlinear second-order elliptic partial differential equation of the magnetic flux function derived independently by Grad and Rubin [1], Shafranov [2] and Lust et al. [3]. This equation is called the Grad–Shafranov equation or the Grad–Schltiter–Shafranov equation.

According to Eq. (7) the right-hand side of the equation includes the pressure and poloidal current profiles both of which depend on poloidal flux, ψ. In some studies the simple pressure and poloidal current profiles are used to convert the GS equation into a linear, inhomogeneous partial differential equation, which is much simpler to solve analytically. These kinds of profiles are not useful physically but have the exact analytical solution and are being used extensively in validating the numerical computations.

In order to test more realistic profiles their dependence on ψ is considered. Therefore the GS equation should be solved iteratively to overcome this nonlinearity. Consequently, according to the described method the following linear problem is solved at kth iteration step,

$$\begin{aligned}&\psi^{k+1}=0,\quad \text{on } \partial\Omega,\\&\Delta^{*}\psi^{k+1}=-\mu_0J_{tr}(R,Z,\psi^{k}),\quad \text{in } \Omega.\end{aligned} \tag{8}$$

where Ω denotes a domain bounded by a closed curve $\partial\Omega$. The