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Space–time localized radial basis function collocation method for solving parabolic and hyperbolic equations

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ABSTRACT

A radial basis collocation method, to solve parabolic and hyperbolic equations, based on the local space–time domain formulation is developed and presented in this paper. The method is different from those that approximate the time derivative using different formulas such as the implicit, explicit, method of lines, or other numerical methods. Considering a partial differential equation with d spatial dimensions, our technique solves the problem as a $(d+1)$-dimensional one without distinguishing between space and time variables, and the collocation points have both space and time coordinates. The parabolic equation is solved using the governing domain equation as a condition on the boundary characterized by the final time T. The hyperbolic equation is solved using two different methods. The first one is based on adapting the technique used for solving parabolic equations. The second one is a new technique that looks at the problem as an ill-posed one with incomplete boundary condition data at the final time T of the space–time domain. The accuracy of our proposed method is demonstrated through different examples in one-, two- and three-dimensional spaces on regular and irregular domains.

Contents

1. Introduction

Initial boundary value problems (parabolic or hyperbolic) can be solved using different numerical methods such as finite

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element, finite volume, boundary element, meshless methods, fundamental solutions, spectral and wavelet methods . In most published works, these methods are based on differentiating between time and space variables. They all start by discretizing the time variable using implicit, explicit, Runge–Kutta or any other known method such as the method-of-line approach, and solve the problem by computing the approximate solution at each time t. Some work was published using space–time finite element method, such as the technique developed by Tazduyar et al. in [8] for the computation of fluid–structure interaction problems. Klaij et al. have also developed the space–time discontinuous Galerkin finite element method for solving compressible Navier-Stokes equations in [9] and advection–diffusion problems in [10]. The technique has also been applied to shallow water flows by Ambati et al. in [11].

To our best knowledge the space–time meshless formulation has been discussed only by a small number of researchers, such as Netuzhylov in [12] where the author developed the space–time meshless collocation method based upon the Interpolating Moving Least Squares (IMLS) technique and applied it to solve coupled problems with moving boundaries. Young et al. [15] have used time-dependent fundamental solutions to solve homogeneous diffusion equations directly. Their proposed scheme can be considered a space–time collocation method as it is free from time discretization. The first published work on the space–time approach using radial basis functions (RBF) is the paper by Li and Mao [5]. They applied the global collocation method known as Kansa method using the Multiquadric function (MQ). The technique developed is used to solve only the parabolic inverse problem of groundwater contaminant source identification. Beside the fact that their formulation is global, the algebraic system obtained is not square and the least squares method was introduced to overcome the ill-posedness of the linear system. Their technique was also used to solve inverse heat conduction problems [4]. Furthermore, Li and Mao developed in [19] a global space–time radial basis collocation model for estimating river pollution sources. In their analysis, they used the RBF developed by Myers et al. in [6] defined as the product of two RBFs, one depending on the space variable and the other on the time variable in the form $\phi(x,t)=\phi_1(x)\phi_2(t)$ where ϕ_1 and ϕ_2 are RBFs on space and time variables, respectively. Their technique is applied to parabolic equations only, see [6].

In this paper we develop an RBF-based space–time localized meshless collocation method based to solve d-dimensional parabolic and hyperbolic problems in $(d+1)$-dimensional space–time domains without differentiating between space and time variables. We study the solution of the following time-dependent equations:

$$\begin{aligned} &\frac{\partial u}{\partial t}(x,t)+Lu(x,t)=f(x,t) \quad \text{in } \Omega\times(0,T]\\ &Bu(x,t)=g(x,t) \quad \text{on } \partial\Omega\times(0,T]\\ &u(x,0)=u_0(x) \quad \text{in } \Omega \end{aligned} \tag{1}$$

and

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2}(x,t)+Lu(x,t)=f(x,t) \quad \text{in } \Omega\times(0,T]\\ &Bu(x,t)=g(x,t) \quad \text{on } \partial\Omega\times(0,T]\\ &u(x,0)=u_0(x) \quad \text{in } \Omega\\ &\frac{\partial u}{\partial t}(x,0)=u_1(x) \quad \text{in } \Omega \end{aligned} \tag{2}$$

without first discretizing the time and then solving the problem in space domain, as it is usually the case with many numerical methods. Here L denotes a second order linear differential operator, B is a boundary operator, f, g, u_0 and u_1 are given smooth functions and Ω is a subset of $\mathbb{R}^d$.

Besides some technical points in the numerical implementation, the main originality of our paper is the use of a local formulation of the RBF collocation method on the space–time domain to solve parabolic and hyperbolic equations. Another originality of our work is the application of the developed technique to solve hyperbolic equations by considering them as ill-posed problems. The developed formulation leads always to a square algebraic system which is not the case in [4,5,19,6]. The main advantages of our technique are

1. removing the need for a discussion of the time stability analysis of the discrete system as it is the case for other time integration techniques, such as the explicit method, θ-method and others,
2. reducing the computational time as there is no need to recompute the matrix for the resulting algebraic system at each time level, unlike the case for others time integration methods used to solve PDEs with time-dependent coefficients.

The paper is organized as follows. In Section 2 we introduce the space–time formulation of parabolic and hyperbolic problems. In Section 3 we give the interpolation by RBFs in space–time domain. In Section 4 we present the space–time localized RBF collocation method with a brief recall of the global RBF collocation method. Section 5 is devoted to the discussion of results obtained by solving different parabolic and hyperbolic examples in one, two, and three-dimensions in both regular and irregular domains. We conclude in Section 6.

2. Space–time problem formulation

In this formulation the traditional d-dimensional time-dependent problem in space is transformed into a $(d+1)$-dimensional one. The problem is then formulated in spatial–temporal variables. The boundary of the new domain $\Omega_T=\Omega\times[0,T]$ given in Fig. 1 is defined by $\partial\Omega\times[0,T]$, $\Omega\times\{t=0\}$ and $\Omega\times\{t=T\}$. The formulation of the technique depends on the type of the problem considered. For the case of a parabolic equation, defined by Eq. (1), we formulated the system of equations as a boundary-value problem in the new space–time domain, with boundary conditions on the boundary of the space–time domain. The system of equations is then written as

$$\frac{\partial u}{\partial t}(x,t)+Lu(x,t)=f(x,t) \tag{3}$$

for the equation in the space–time domain $\Omega\times(0,T)$, and

$$\begin{aligned} &Bu(x,t)=g(x,t)\\ &u(x,t)=u_0(x) \end{aligned} \tag{4}$$

on $\partial\Omega\times[0,T]$ and $\Omega\times\{t=0\}$, respectively. As the problem is still ill-posed for the space–time domain since it needs a boundary condition on $\Omega\times\{t=T\}$, Eq. (3) can be considered as a boundary condition on $\Omega\times\{t=T\}$

$$\frac{\partial u}{\partial t}(x,t)+Lu(x,t)=f(x,t) \quad \text{on } \Omega\times\{t=T\}. \tag{5}$$

The parabolic problem is then summarized as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t)+Lu(x,t) &= f(x,t) \quad \text{in } \Omega\times(0,T]\\ Bu(x,t) &= g(x,t) \quad \text{on } \partial\Omega\times(0,T]\\ u(x,t) &= u_0(x) \quad \text{on } \Omega\times\{t=0\}\\ \dfrac{\partial u}{\partial t}(x,t)+Lu(x,t) &= f(x,t) \quad \text{on } \Omega\times\{t=T\} \end{cases} \tag{6}$$

The new formulation leads to a well-posed problem that can be solved once to approximate the solution at any point (x,t). Note that the linear algebraic system is square since the number of