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Radial basis function collocation method for an elliptic problem with nonlocal multipoint boundary condition

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ABSTRACT

Radial basis function domain-type collocation method is applied for an elliptic partial differential equation with nonlocal multipoint boundary condition. A geometrically flexible meshless framework is suitable for imposing nonclassical boundary conditions which relate the values of unknown function on the boundary to its values at a discrete set of interior points. Some properties of the method are investigated by a numerical study of a test problem with the manufactured solution. Attention is mainly focused on the influence of nonlocal boundary condition. The standard collocation and least squares approaches are compared. In addition to its geometrical flexibility, the examined method seems to be less restrictive with respect to parameters of nonlocal conditions than, for example, methods based on finite differences.

1. Introduction

Mathematical models arising in various fields of science and engineering (for example, thermoelasticity [1], thermodynamics [2], hydrodynamics [3], biological fluid dynamics [4] or plasma physics [5]) are very often expressed in terms of partial differential equations (PDEs) and nonclassical constraints, which are usually identified as *nonlocal* (*boundary*) *conditions*. As a rule, the appearance of nonlocal conditions makes quite a number of theoretical and numerical challenges. Therefore, nonlocal differential problems receive a lot of attention in the literature. Many studies are dedicated to various aspects of the numerical solution of such nonclassical problems.

Finite differences, finite elements or finite volumes are examples of discretisation techniques which are widely and successfully applied to approximate solution of various PDEs. These traditional approaches are based on the domain discretisation using mesh. The mesh generation can be quite a difficult task in the case of three-dimensional domains with complex shapes. One of the ways to overcome problems related to the meshing are the so-called *meshless methods*, which gained a lot of attention in recent years (see e.g. [6–8]).

In this paper, for the solution of a model problem with nonlocal boundary condition we apply a meshless discretisation technique based on the *radial basis functions* (RBFs) [9–14]. In papers [15–19], RBFs were already used for the spatial discretisation of time dependent (parabolic and hyperbolic) equations with nonlocal integral conditions. Experiments with various test examples have demonstrated that RBF-based collocation methods can be successfully applied to solve such a kind of nonlocal problems. Recently, the method of approximate particular solutions using multiquadric (MQ) RBFs has been applied to the time-fractional diffusion equation with nonlocal boundary condition [20].

It is well-known that properties of the numerical methods for nonlocal differential problems usually depend on the parameters of nonlocal boundary conditions. For example, the stability of finite difference schemes are related to the spectral properties of certain matrices and, in the case of nonlocal problems, these properties depend on the parameters appearing in nonlocal conditions [21–23]. Therefore, numerical methods for the solution of PDEs with nonlocal conditions require special attention. The influence of nonlocal conditions on the properties of any numerical method which is applied to solve a certain nonlocal problem should always be investigated very carefully.

In paper [24], RBF collocation technique was applied to solve a model problem for two-dimensional Poisson equation on the unit rectangle. Two-point or integral condition was formulated on one side of the rectangle. Besides the standard testing of the method, the influence of nonlocal conditions on the optimal selection of the RBF shape parameter, as well as on the conditioning and

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accuracy of the method was investigated. Later, the same problem has been solved in paper [25], where collocation methods based on Haar wavelets and RBFs have been applied. RBF-based collocation technique was also used to solve a multidimensional elliptic equation with nonlocal integral conditions [26]. The influence of the RBF shape parameter and distribution of the nodes on the accuracy of the method as well as the influence of nonlocal conditions on the conditioning of the collocation matrix were investigated by analysing two- and three-dimensional test problems with the manufactured solutions.

In the present work we continue our investigation and consider a model problem which consists of Poisson equation with mixed boundary conditions. One of these conditions is *nonlocal multipoint boundary condition* relating the boundary values of unknown function to several values inside the domain. A meshless method for the solution of such problem allows us to eliminate connection between the domain discretisation and points defining nonlocal part of the multipoint boundary condition. To the best of our knowledge, this is the first time when an elliptic PDE with nonlocal multipoint boundary condition is solved using an RBF-based meshless method.

The main aim of this work is to investigate the properties of the method and, in particular, their dependence on the parameters of nonlocal condition. We do not provide any theoretical results. Instead, we conduct an extensive numerical study. The insights made from the numerical study can help us gain the basic understanding of the properties of the method. It should be mentioned that, when dealing with nonlocal problems, numerical studies (computational experiments) are often utilised as research methods even when such classical and well-established techniques as finite differences are applied (see e.g. [22,27]).

The paper is organised as follows. In Section 2, we give a detailed formulation of the model problem and some additional related references. A meshless method based on RBF collocation is described in Section 3. By analysing a test example, the method is investigated in Section 4. Finally, Section 5 concludes the paper with summarising remarks and possible directions for the future research.

2. Model problem

A model problem considered in this paper consists of Poisson equation and mixed boundary conditions:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, & \text{(1a)}\\ u = g & \text{on } \Gamma_1, & \text{(1b)}\\ u = \sum_{\mathbf{x}_l^* \in \Omega^*} \gamma_l u(\mathbf{x}_l^*) + h & \text{on } \Gamma_2, & \text{(1c)} \end{cases}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is a bounded domain, $\partial\Omega = \Gamma_1 \cup \Gamma_2$ (with $\Gamma_1 \cap \Gamma_2 = \varnothing$ and $\Gamma_2 \neq \varnothing$), f, g and h are given functions. While Dirichlet condition (1b) is an example of classical boundary condition, the condition (1c) is nonlocal and relates the values of the solution u on the boundary part Γ_2 to the values at the interior points $\mathbf{x}_l^* \in \Omega^* \subset \Omega$ (subset Ω^* is discrete). The nonlocal condition (1c) is defined by parameters γ_l and $\mathbf{x}_l^*$. The weights γ_l can be either constants, or functions ($\gamma_l = \gamma_l(\mathbf{x})$), or values of the functions at $\mathbf{x}_l^* \in \Omega^*$ ($\gamma_l = \gamma_l(\mathbf{x}_l^*)$). When $\gamma_l \equiv 0$, condition (1c) becomes Dirichlet boundary condition. If condition (1c) relates the boundary values to a single interior point ($|\Omega^*| = 1$), it is usually referenced to as *Bitsadze–Samarskii nonlocal condition*.

Nonlocal multipoint boundary conditions are related to nonlocal integral conditions [21]. For instance, the multipoint condition (1c) is a special case of nonlocal integral condition

$$u = \int_\Omega \gamma(\mathbf{x}) u(\mathbf{x}) \mathrm{d}\mathbf{x} + h \quad \text{on } \Gamma_2.$$

Indeed, we get the nonlocal multipoint boundary condition (1c) when the weight function is defined as

$$\gamma(\mathbf{x}) = \sum_{\mathbf{x}_l^* \in \Omega^*} \gamma_l \delta(\|\mathbf{x} - \mathbf{x}_l^*\|_2),$$

where δ is the Dirac delta function.

Numerical methods for the solution of PDEs with nonlocal discrete boundary conditions have been considered in many papers. For example, we can mention papers related to the approximate solution of nonlocal problems for elliptic [28–31], elliptic–parabolic [32], hyperbolic [33,34], or hyperbolic–parabolic [35] equations with multipoint or Bitsadze–Samarskii nonlocal boundary conditions. The paper [36] presents an efficient way of implementing general multipoint constraint conditions arising in finite element analysis related to structural mechanics. In paper [37], a two-dimensional reaction–diffusion problem with Bitsadze–Samarskii nonlocal boundary condition was solved using meshless local Petrov–Galerkin (MLPG) method and MQ RBFs were used for the spatial discretisation of local weak equations.

3. Construction of the method

3.1. A brief overview of radial basis functions

RBFs already proved to be quite an effective tool both for the scattered data interpolation [10] and approximate solution of PDEs [11–14]. We give a very brief introduction to RBFs. More details can be found in books [9,10,12,14]. The book [12] also reviews the latest advances on RBF collocation methods for the numerical solution of PDEs.

A multivariate real-valued function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is called a *radial function* if there exists a univariate function $\phi : [0, \infty) \to \mathbb{R}$ such that

$$\Phi(\mathbf{x}) = \phi(\|\mathbf{x}\|),$$

where $\|\cdot\|$ is some norm on $\mathbb{R}^d$ (usually the Euclidean norm is used). That is, the function $\Phi(\mathbf{x})$ can be expressed in the Euclidean distance variable $r = \|\mathbf{x}\|$.

Table 1 gives several examples of widely used RBFs. All these RBFs are globally supported and infinitely smooth. The shape (flatness) of each given RBF is controlled by a positive parameter ϵ which is called the *shape parameter*. Equivalently, a reciprocal shape parameter $c = 1/\epsilon$ also can be used in the expressions of the same functions.

In the case of an interpolation problem, the positive definiteness of RBF is an important property which ensures the invertibility of the interpolation matrix. IMQ, IQ and GA RBFs are strictly positive definite, while MQ RBF is conditionally positive definite of order one [9,10]. The conditional positive definiteness means that the invertibility of the interpolation problem is ensured by adding a polynomial of a certain order to the interpolant and by augmenting the interpolation system with some additional equations. Since the polynomial augmentation can increase the condition number of the interpolant matrix but not necessary the accuracy of the interpolation [38], and singular cases of the interpolation

Table 1
Globally supported and infinitely smooth RBFs ($r = \|\mathbf{x}\|$).

RBF	Definition
Multiquadric (MQ)	$\phi(r) = \sqrt{1 + (\epsilon r)^2}$
Inverse multiquadric (IMQ)	$\phi(r) = (\sqrt{1 + (\epsilon r)^2})^{-1}$
Inverse quadric (IQ)	$\phi(r) = (1 + (\epsilon r)^2)^{-1}$
Gaussian (GA)	$\phi(r) = \exp(-(\epsilon r)^2)$