Engineering Analysis with Boundary Elements 64 (2016) 230–236

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Improved finite integration method for partial differential equations

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ARTICLE INFO

Article history:
Received 14 September 2015
Received in revised form
13 December 2015
Accepted 16 December 2015
Available online 9 January 2016

Keywords:
Finite integration method
Numerical quadrature
Simpson's rule
Cotes integral formula
Lagrange interpolation

ABSTRACT

Based on the recently developed finite integration method (FIM) for solving one-dimensional partial differential equations by using the trapezoidal rule for numerical quadrature, we improve in this paper the FIM with an alternative extended Simpson's rule in which the Cotes and Lagrange formulas are used to determine the first order integral matrix. The improved one-dimensional FIM is then further extended to solve two-dimensional problems. Numerical comparison with the finite difference method and the FIM (Trapezoidal rule) are performed by several one- and two-dimensional real application including the Poisson type differential equations and plate bending problems. It has been shown that the newly revised FIM has made significant improvement in terms of accuracy compare without much sacrifice on the stability and efficiency.

1. Introduction

Partial differential equations (PDEs) commonly appear in mathematical modeling to describe a wide variety of physical phenomena such as fluid and solid mechanics. The properties and characteristics of the physical phenomena can then be understood from the closed form solutions to these PDEs. However, under various boundary conditions and real problem configuration, it is very rare that these models can be solved in closed form solutions. Due to the advancement of computational methods, numerical approximations can usually be achieved inexpensively to give high accuracy together with a reliable bound on the error between the analytical solution and its numerical approximation. There are many numerical techniques available for solving differential equations [1–5] among which the finite difference method (FDM), Finite Element Method (FEM) and Boundary Element Method (BEM) are commonly used.

Recently, Wen et al. [6] and Li et al. [7,8] developed a new finite integration method (FIM) for solving one- and two-dimensional partial differential equations and successfully demonstrated its applicability for solving nonlocal elasticity problems. It has been shown that the FIM gives a much higher degree of accuracy than the FDM and the Point Collocation Method (PCM). In this paper, an improved FIM is developed by using an alternative extended Simpson's rule, Cotes integral formula, and Lagrange formula for solving one- and two-dimensional partial differential equations. Similar to the FDM and the PCM, a finite number of points, known as field points, are distributed in the computational domain. The field points are generated either uniformly (grid) along the independent coordinate or randomly distributed in the domain. The integration matrix of the first order is obtained by direct integration with Simpson's rules, Cotes formula, and Lagrange formula. Based on these first order integration matrices, the multi-layer finite integration matrix can easily be obtained. To demonstrate the accuracy and efficiency of the improved FIM, several one-dimensional and two-dimensional numerical examples are given and compared with the FDM and analytical solution.

2. Finite integration method for one dimension

2.1. Trapezoidal rule (TR)

A simple computational scheme for integration was introduced in [6,7], which was called an Ordinary Linear Approach (OLA) as

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http://dx.doi.org/10.1016/j.enganabound.2015.12.012
0955-7997/

follows. Let

$$U(x_k)=\int_0^{x_k}u(\xi)d\xi\cong\sum_{i=1}^{k}a_{ki}u(x_i). \tag{1}$$

For the most simple trapezoidal rule, the coefficients are

$a_{1i}=0,$

$$a_{ki}=\begin{cases}0.5h, & i=1,\\ h & i=2,3,...,k-1,\\ 0.5h & i=k,\\ 0 & i>k,\end{cases} \tag{2}$$

where $x_i=(i-1)h, h=b/(N-1), i=1,2,...,N$ are nodal points in [0, b], and $x_1=0, x_N=b$. Note that (1) can be written in matrix form as

$$\mathbf{U}=\mathbf{Au} \tag{3}$$

where $\mathbf{U}=[U_1,U_2,...,U_N]^T$, $\mathbf{u}=[u_1,u_2,...,u_N]^T$,

$$\mathbf{A}=(a_{ki})=h\begin{pmatrix}0&0&0&0&0&0\\1/2&1/2&0&0&0&0\\1/2&1&1/2&0&0&0\\1/2&1&1&1/2&0&0\\...&...&...&...&...&...\\1/2&1&1&1&1&1/2\end{pmatrix}_{N\times N} \tag{4}$$

is the first order integration matrix in which $U_i=U(x_i), u_i=u(x_i)$ denote the values of integration and the integral function, respectively, at each node. The single-integral (1) can be extended to a multi-integral for one-dimensional problems as follow:

$$U^{(2)}(x)=\int_0^x\int_0^\zeta u(\xi)d\xi d\zeta,\quad x\in[0,b]. \tag{5}$$

Applying the OLA technique again for the integral function $U^{(2)}(x)$, we have

$$U^{(2)}(x_k)=\int_0^{x_k}\int_0^\zeta u(\xi)d\xi d\zeta\cong\sum_{i=0}^{k}\sum_{j=0}^{i}a_{ki}a_{ij}u(x_i)=\sum_{i=0}^{k}a_{ki}^{(2)}u(x_i). \tag{6}$$

From (3), the multi-integral (6) can be written in matrix form as

$$\mathbf{U}^{(2)}=\mathbf{A}^{(2)}\mathbf{u}=\mathbf{A}^2\mathbf{u} \tag{7}$$

where

$$\mathbf{A}^{(2)}=\mathbf{AA}=h^2\begin{pmatrix}0&0&0&0&0&0\\1/4&1/4&0&0&0&0\\3/4&1&1/4&0&0&0\\5/4&2&1&1/4&0&0\\...&...&...&...&...&...\\ [1+2(N-2)]/4&N-2&N-3&...&1&1/4\end{pmatrix}_{N\times N} \tag{8}$$

and the elements of matrix $\mathbf{A}^{(2)}$ are given by [6,7]

$a_{1i}^{(2)}=0$

$$a_{ki}^{(2)}=\begin{cases}[1+2(k-2)]h^2/4, & i=1,\\ (k-i)h^2, & i=2,3,...,k-1,\\ h^2/4, & i=k,\\ 0, & i>k.\end{cases} \tag{9}$$

2.2. Alternative extended Simpson rule

The numerical accuracy of the first order integral matrix $\mathbf{A}$ can be further improved by using the alternative extended Simpson's rule. The simplest trapezoidal rule for the integration at the second node $x_2=h$ is

$$U(x_2)=\int_0^h u(x)dx=h\left(\frac{1}{2}u_1+\frac{1}{2}u_2\right). \tag{10}$$

In order to improve the above integral accuracy, we consider the three node Lagrange interpolation for u in (10), i.e.

$$u(x)=\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}u_1+\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}u_2+\frac{(x-x_1)(x-x_2)}{(x_3-x_2)(x_3-x_1)}u_3. \tag{11}$$

Substituting (11) into (10), we have

$$U(x_2)=\int_0^h u(x)dx\cong h\left(\frac{5}{12}u_1+\frac{2}{3}u_2-\frac{1}{12}u_3\right). \tag{12}$$

By the Simpson integral rule and [9], we have

$$U(x_3)=\int_0^{2h}u(x)dx\cong h\left(\frac{1}{3}u_1+\frac{4}{3}u_2+\frac{1}{3}u_3\right), \tag{13}$$

$$U(x_4)=\int_0^{3h}u(x)dx\cong h\left(\frac{3}{8}u_1+\frac{9}{8}u_2+\frac{9}{8}u_3+\frac{3}{8}u_4\right) \tag{14}$$

for nodes $i\le 4$. This extended formula of order $1/N^3$ gives

$$U(x_i)=\int_0^{(i-1)h}u(x)dx\cong h\left(\frac{5}{12}u_1+\frac{13}{12}u_2+u_3+u_4+...u_{i-2}+\frac{13}{12}u_{i-1}+\frac{5}{12}u_i\right) \tag{15}$$

for i=5, 6, …, $N+1$.
Thus, the coefficients in (1) are given as

$$a_{1i}=0\quad i=1,2,...,N \tag{16a}$$

$$a_{21}=\frac{h}{2}, a_{22}=\frac{h}{2},\ \text{(Simpson I)} \tag{16b}$$

or

$$a_{21}=\frac{5h}{12}, a_{22}=\frac{2h}{3}, a_{23}=-\frac{h}{12},\ \text{(Simpson II)} \tag{16c}$$

$$a_{31}=\frac{h}{3}, a_{32}=\frac{4h}{3}, a_{33}=\frac{h}{3}, \tag{16d}$$

$$a_{41}=\frac{3h}{8}, a_{42}=\frac{9h}{8}, a_{43}=\frac{9h}{8}, a_{44}=\frac{3h}{8}. \tag{16e}$$

For all nodes $i>4$, the alternative extended Simpson's rule is represented as [9]:

$$a_{i1}=\frac{5h}{12}, a_{i2}=\frac{13h}{12}, a_{i3}=h, ..., a_{i(i-2)}=h, a_{i(i-1)}=\frac{13h}{12}, a_{ii}=\frac{5h}{12} \tag{16f}$$

Note that the first order integral matrix $\mathbf{A}$ using the extended Simpson's rule is diagonal except the second row for Simpson II. In addition, the multi-integral formula (7) is still valid for the improved integration matrix using the extended Simpson's rule, i.e. $\mathbf{A}^{(2)}=\mathbf{A}^2$. To satisfy the boundary conditions, we may need higher order derivatives which can be achieved by considering the four node interpolation at the left-end point $x_1=0$ to obtain the first and second orders derivatives as follows

$$\left.\frac{du}{dx}\right|_{x=0}=\frac{(x_2x_3+x_3x_4+x_4x_2)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}u_1+\frac{(x_1x_3+x_3x_4+x_4x_1)}{(x_2-x_3)(x_2-x_4)(x_3-x_1)}u_2+\frac{(x_1x_2+x_2x_4+x_4x_1)}{(x_3-x_4)(x_3-x_1)(x_3-x_2)}u_3+\frac{(x_1x_2+x_2x_3+x_3x_1)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}u_4$$
$$=-\frac{11}{6h}u_1+\frac{3}{h}u_2-\frac{3}{2h}u_3+\frac{1}{3h}u_4, \tag{17}$$