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A new approximate method for an inverse time-dependent heat source problem using fundamental solutions and RBFs

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ABSTRACT

This paper presents a meshless numerical scheme to solve the inverse heat source time dependent problem. Fundamental solutions of heat equations and radial basis functions (RBFs) are used to obtain a numerical solution. Since the coefficient matrix may be ill-conditioned, the Tikhonov regularization (TR) method is employed to solve the resulting system of linear equations. Therefore, the generalized cross-validation (GCV) criterion is applied to choose a regularization parameter. The accuracy and efficiency of the proposed method is illustrated by some numerical examples.

1. Introduction

Transport, diffusion and conduction are important natural processes defined by the following parabolic partial differential equation:

$$u_t - a^2 \Delta u = f(\mathbf{x}, t; u), \quad (\mathbf{x}, t) \in \Omega \times (0, t_{\max}) \tag{1}$$

where u is the state variable, Ω is a bounded domain in $\mathbb{R}^d$, a is the diffusion coefficient, Δ is the Laplace operator and f is the physical source term. For a given function f, Eq. (1) with initial and boundary conditions is a direct (or forward) problem that is known to be well-posed. We must know the heat source characteristics to determine the temperature of conduction in many practical problems. In many cases, the characteristics of the heat source are often unknown. To overcome this difficulty, we usually determine the heat source from a measured temperature at a fixed location, x_0 (or at a particular time, t^*) of the solution domain. This kind of problem is called the identification problem of unknown source term and it is both an inverse and ill-posed problem. The problem referred to as an inverse heat source problem has unknown parameters or boundary conditions. Such problems occur in many engineering branches. Since the existence, uniqueness and stability of the solutions of these problems are not usually guaranteed, they are generally defined as ill-posed problems [1,2]. The inverse problem of determining an unknown heat source function in the heat conduction equation is one of the most important inverse problems, appearing in some engineering sciences that has been studied by researchers as Cannon [3,4] and Fatullayev [5]. They have analyzed the inverse source problem with the Dirichlet–Neumann conditions in which $f = f(\mathbf{x}, t; u)$. In the current paper, the heat source is a function of time only. The existence and uniqueness of solutions of these problems have been verified by Savateev [6]. In his work, the source term has been defined as a function of both time and space variables that are separable. Moreover, some scientists have proposed that the heat source is a function of $\mathbf{x}$ or t. In [8,9,11–14,21], several numerical schemes have been proposed to determine a space-dependent heat source. In [7,22], the method of fundamental solutions has been presented for an inverse time-dependent heat source problem. In [44–46], these numerical methods were considered to solve some two and three-dimensional inverse transient heat source problems. Although many different numerical schemes have been introduced in recent decades to solve the inverse heat source problem, the inverse heat problem has not completely been solved yet. Several numerical methods have been presented to solve the inverse source problem. Among them are those based upon finite differences (FDM)[10], finite element method (FEM) and the boundary element method (BEM) [11,12] are some of the traditional methods for solving the inverse source problem. Furthermore, other methods such as linear least-squares

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error method, sequential method and iterative regularization methods [13–15] have been used to solve the inverse source problem. For all these methods, the differential equations need to be discretized. The traditional methods like FDM and FEM require meshing of the solution space; are highly time consuming to compute the solutions. However, the ill-posing of these kinds of problems and ill-conditioning of the resulting discretized matrix are the main problems in constructing numerical algorithms for determining these problems. Studies related to the definitions of conditional stability of results have been presented in [16]. Additionally, within recent years, many scientists in the field of applied sciences and engineering have considered meshless methods. Radial basis functions (RBFs) method [17–19] and the method of fundamental solutions (MFS) [7,20–22] are amongst the meshless methods that can be counted as the alternative methods of mesh-dependent ones unlike FDM and FEM. In meshless methods, not only the computation efficiency increases but also they are simply applied to solve differential equations in higher dimensions and can be applied to non-smooth and non-regular domains.

Hardy [23] introduced a method in which a bivariate interpolant was needed to represent the exact position and construct contours from sparse, scattered data [24,25]. In 1990s, Kansa applied RBFs method to solve different types of partial differential equations [26,27]. Different types of mathematical problems from partial or ordinary differential equations to integral equations have been solved by Kansa and some other scientists [28–31]. Kupradze and Alecsidze [32] first introduced MFS which is a relatively new meshless scheme to solve certain boundary value problems. MFS defines the solution of the problem as a linear combination of fundamental solutions. Since this method is originally meshless, it is applied to solve different types of partial differential equations [7–9,33–37]. In recent years, this method has been used to solve inhomogeneous problems. Hon [38], Mera [39] and Marin [40,41] applied the MFS to solve the inverse heat conduction problem (IHCP) and the Cauchy problem associated with 2D Helmholtz-type equations. Liang et al. [7,8] utilized this method to solve some of inverse source problems. The MFS produces fully populated matrices, therefore the main disadvantage of this method is that the system of equations could be ill-conditioned even for some well-posed problems. Recently, new methods have been developed by making some of changes to the RBFs and MFS. Ming et al. [42] presented a meshless method for solving non-homogeneous Cauchy problems using MFS and the method of particular solution (MPS). Wei et al. [43] introduced the adaptive cross approximation (ACA) to accelerate the MFS in large scale potential problems. Wang et al. [44] proposed a new general numerical method for solving an inverse boundary value problem using the MFS in combination with the dual reciprocity method (DRM). The identification of heat source dependent in space and time was considered by Mierzwiczak and Kolodziej [45] using the MFS with the discretization of time derivative by θ-method. Also, Mierzwiczak and Kolodziej [46] presented a version of the MFS with the Laplace transformation for solving the inverse transient heat source problem. Recently, in addition to the above-mentioned methods, Xiong et al. [47,48] investigated an inverse one-dimensional heat conduction problem in a multilayer domain via an effective regularization method based on Laplace and Fourier transform techniques. Furthermore, a variety of novel boundary-type methods have been introduced [49–51]. Accordingly, this paper intends to provide a new method based upon combining radial basis functions and fundamental solutions of the heat equation in order to solve the inverse heat source problem depending on the time parameter. Our objective is to combine RBFs and the MFS to obtain a more accurate method to solve the inverse problem at the beginning and ending points of the spatial interval in comparison with only MFS. To do this, we change the inverse problem into a problem with only one unknown by transformations, and then we introduce the method of fundamental solutions-radial basis functions (MFS-RBF), and we solve it for the resulting direct problem. Using MQ-RBFs, the numerical results depend on choosing an appropriate value of the shape parameter [52]. Selecting a proper value of the shape parameter can increase the accuracy of the numerical results. To counteract ill-conditioning using finite precision arithmetic in discretizing the problem with our method, we applied the Tikhonov regularization (TR) method in order to solve the system of the linear equations. The generalized cross-validation (GCV) criterion has been used to find an optimum value of the regularization parameter. There are some examples to demonstrate the efficiency of the present method to MFS.

The paper is organized as follows: Section 2 presents the mathematical formulation of inverse source problem. Section 3 introduces the method of fundamental solutions (MFS) and radial basis functions (RBFs) method. Section 4 applies the method of fundamental solutions-radial basis functions (MFS-RBF). Section 5 presents the Tikhonov regularization method. Section 6 reveals some numerical examples including continuous and discontinuous functions. Section 7 concludes the paper and gives some suggestions.

2. Mathematical formulation of the problem

We consider the one-dimensional problem in which the source term $f(x,t;u)=f(t)$ depends on just time parameter and satisfies:

$$u_t = a^2 u_{xx} + f(t), \quad (x,t) \in \Omega = (0,l) \times (0,t_{\max}] \tag{2}$$

with the following initial and boundary conditions:

$$u(x,0) = u_0(x), \quad x \in [0,l], \tag{3}$$

$$u(0,t) = p(t), \quad t \in [0,t_{\max}], \tag{4}$$

$$u(l,t) = q(t), \quad t \in [0,t_{\max}], \tag{5}$$

and the extra condition:

$$u(x_0,t) = h(t), \quad t \in [0,t_{\max}], \tag{6}$$

in which $x_0 \in (0,l)$, $u_0(x), q(t)$ and $h(t)$ are given functions satisfying the below compatibility conditions:

$$u_0(0) = p(0), \quad u_0(l) = q(0), \quad u_0(x_0) = h(0). \tag{7}$$

Problems of this type include inverse problems, heat conduction processes, hydrology, material sciences and heat transfer problems. In the context of heat conduction and diffusion, u represents either temperature or concentration. The unknown $f(t)$ is interpreted as either a heat or material source, respectively in a chemical or a biochemical application, $f(t)$ may be interpreted as a reaction term [53]. Under an additional a priori condition, the unique solvability of the inverse problem (2)–(7) can be obtained (see Theorem 1 in [12]). The existence and uniqueness of solutions to similar problems have been studied in [6,12]. Nevertheless, since the solution does not depend continuously on the input data, it is considered as an ill-posed problem. Using the following transformations [7]:

$$r(t) = \int_0^t f(\xi)\, d\xi, \tag{8}$$

$$v(x,t) = u(x,t) - r(t), \tag{9}$$

The problem (2)–(6) is modified into the following problem:

$$v_t = a^2 v_{xx}, \quad (x,t) \in (0,l) \times (0,t_{\max}], \tag{10}$$

$$v(x,0) = u_0(x), \quad x \in [0,l], \tag{11}$$