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Boundary methods for mixed boundary problems of Laplace's equation in elliptic domains with elliptic holes ☆

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ABSTRACT

The particular solutions (PS) and fundamental solutions (FS) in polar coordinates can be found in many textbooks, but with much less coverage in elliptic coordinates (Chen et al., 2010 [5], Chen et al., 2012 [6], Morse and Feshbach, 1953 [20], Li et al., 2015 [18]). Since the elliptic domains with elliptic holes may be found in some engineering problems, the PS and the FS expansions in elliptic coordinates are *essential* for numerical computations. For Dirichlet problems of Laplace's equation in elliptic domains, the null field method (NFM), the interior field method (IFM) and the collocation Trefftz method (CTM) are reported in [18]. There seems to exist few reports for mixed problems, where the Dirichlet and Neumann conditions are assigned on the exterior and the interior boundaries, simultaneously. This paper is devoted to such mixed problems by the NFM and the IFM, and the explicit algebraic equations are derived for elliptic domains. Besides, other effective particular solutions (PS) are sought, and the collocation Trefftz method (CTM) [16] is employed. The CTM may be used for Robin problems in elliptic domains. The effective algorithms for the mixed problems of Laplace's equation on elliptic domains are the main goal of this paper. The techniques of the mixed techniques in this paper can be applied to Dirichlet problems, the dual techniques are called in Chen and Hong (1999 [4]), Hong and Chen (1988 [8]), and Portela et al. (1992 [21]). A preliminary study for the dual techniques is one goal of this paper.

1. Introduction

For Dirichlet problems of Laplace's equation in elliptic domains with elliptic holes, the null field method (NFM), the interior field method (IFM) and the collocation Trefftz method (CTM) are reported in [18]. There seems to exist few reports for mixed problems, where the Dirichlet and Neumann conditions are assigned on the exterior and the interior boundaries, simultaneously. This paper is devoted to numerical algorithms for the mixed problems in elliptic domains with elliptic holes, and the explicit algebraic equations are derived for easy computation. Since the derivatives of the solutions are involved, more analytic work is needed.

The interior field method (IFM) is equivalent to the NFM, when the field nodes are located just on the domain boundary. In fact, the NFM is replaced by the IFM in real computation, see [15]. Besides, the collocation Trefftz method (CTM) is also employed by using the simpler particular solutions in elliptic coordinates. The CTM can be used for the Robin problems, but both the NFM and the IFM suffer from some difficulty. Numerical experiments are carried out for elliptic domains with one elliptic hole by the IFM, the NFM and the CTM.

Let us mention some related work. For Laplace's equation in circular and elliptic domains, the NFM was reported in Chen et al. [3–7]. Recently, the NFM and the IFM are studied for Dirichlet problems in [11,12,15] and for the Neumann problems in [13] for circular domains. Moreover, the NFM, the IFM and the CTM are developed for Dirchlet problem in elliptic domains in [18], nevertheless many more references on circular domains are given.

This paper is organized as follows. In the next section, the basic algorithms of the NFM and their explicit equations are introduced. In Section 3, the explicit equations of second kind of field equations are derived, and the explicit algorithms of the NFM are provided. In Sections 4 and 5, the interior field method (IFM) and the collocation Trefftz method (CTM) are developed, respectively. In Section 6, numerical experiments are carried out for elliptic

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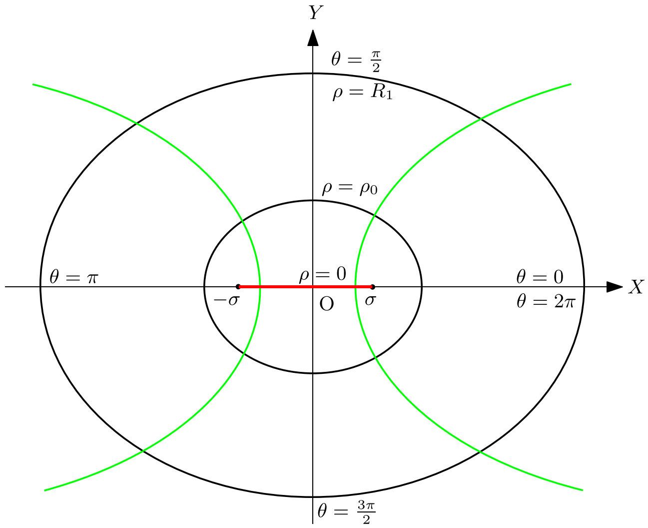

Fig. 1. Elliptic coordinates with (ρ, θ).

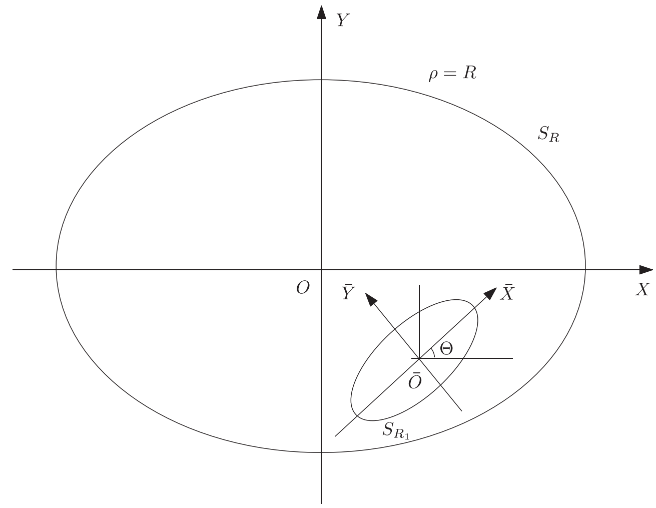

Fig. 2. The ellipse S_R with an elliptic hole S_{R_1}.

domains with one elliptic hole. In the last section, a few conclusions are made.

2. The null field method

2.1. Elliptic coordinates

The elliptic coordinates are defined in [18,20] by (see Fig. 1)

$$x = \sigma \cosh \rho \cos \theta, \quad y = \sigma \sinh \rho \sin \theta, \tag{2.1}$$

where $\sigma > 0$ and two coordinates (ρ, θ) have the ranges $0 \le \rho < \infty$ and $0 \le \theta \le 2\pi$. When $\rho = \rho_0 \in [0, \infty)$, Eq. (2.1) leads to an ellipse

$$\frac{x^2}{\sigma^2 \cosh^2 \rho_0} + \frac{y^2}{\sigma^2 \sinh^2 \rho_0} = 1, \tag{2.2}$$

with two semi-axes $a = \sigma \cosh \rho_0$ and $b = \sigma \sinh \rho_0$.

For explicit algebraic equations, we need the coordinate transformations between different elliptic coordinates. In general, the axes of different ellipsis are not along the x and y axes. The other Cartesian coordinates $\overline{XOY}$ are located from the standard Cartesian coordinates XOY by rotating a counter-clockwise angle $\Theta \in [0, \pi)$, [1] see Fig. 2. Denote the other elliptic coordinates $(\overline{\rho}, \overline{\theta})$ in $\overline{XOY}$

$$\overline{x} = \sigma_1 \cosh \overline{\rho} \cos \overline{\theta}, \quad \overline{y} = \sigma_1 \sinh \overline{\rho} \sin \overline{\theta}, \tag{2.3}$$

where $\sigma_1 > 0$. We cite from [18] the explicit formulas of the transformations for two different elliptic coordinates. The transformation from (ρ, θ) of (2.1) to $(\overline{\rho}, \overline{\theta})$ is given by

$$T : \{(\rho, \theta) \to (\overline{\rho}, \overline{\theta})\}, \tag{2.4}$$

where

$$\overline{\rho} = \text{arc}\sinh(F(\overline{x}, \overline{y}; \sigma_1)), \quad \overline{\theta} = \arccos\left(\frac{\overline{x}}{\sigma_1 \cosh \overline{\rho}}\right), \tag{2.5}$$

$$F(x, y; \sigma) = \frac{1}{\sqrt{2}\sigma} \sqrt{(x^2 + y^2 - \sigma^2) + \sqrt{(x^2 + y^2 - \sigma^2)^2 + 4\sigma^2 y^2}}, \tag{2.6}$$

$$\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} \cos \Theta & \sin \Theta \\ -\sin \Theta & \cos \Theta \end{pmatrix} \begin{pmatrix} \sigma \cosh \rho \cos \theta - x_1 \\ \sigma \sinh \rho \sin \theta - y_1 \end{pmatrix}. \tag{2.7}$$

The inverse transformation from $(\overline{\rho}, \overline{\theta})$ to (ρ, θ) is given by

$$T^{-1} : \{(\overline{\rho}, \overline{\theta}) \to (\rho, \theta)\}, \tag{2.8}$$

where

$$\rho = \text{arc}\sinh(F(x, y; \sigma; x_1, y_1)), \quad \theta = \text{arc}\cos\left(\frac{x - x_1}{\sigma \cosh \rho}\right), \tag{2.9}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \Theta & -\sin \Theta \\ \sin \Theta & \cos \Theta \end{pmatrix} \begin{pmatrix} \sigma_1 \cosh \overline{\rho} \cos \overline{\theta} \\ \sigma_1 \sinh \overline{\rho} \sin \overline{\theta} \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}. \tag{2.10}$$

2.2. Basic algorithms

Denote the large ellipse S_R with $\rho = R$, where the elliptic coordinates (ρ, θ) are given by (2.1) with the origin $(0, 0)$. Also denote a small ellipse $S_{R_1} \subset S_R$ with $\overline{\rho} = R_1$, where the elliptic coordinates $(\overline{\rho}, \overline{\theta})$ are given by

$$\overline{x} = \sigma_1 \cosh \overline{\rho} \cos \overline{\theta}, \quad \overline{y} = \sigma_1 \sinh \overline{\rho} \sin \overline{\theta}, \tag{2.11}$$

where $\sigma_1 > 0$. This Cartesian system $(\overline{x}, \overline{y})$ with the origin (x_1, y_1) is rotated from the axis x, by a counter-clockwise angle Θ as in Fig. 2. The coordinate transformations between (ρ, θ) and $(\overline{\rho}, \overline{\theta})$ are given in (2.4) and (2.8). Denote the solution domain by $S = S_R / S_{R_1}$ and its boundary by $\partial S = \partial S_R \cup \partial S_{R_1}$. In this paper, consider the mixed problem of Dirichlet and Neumann conditions,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in } S, \tag{2.12}$$

$$u_\nu|_{\partial S_R} = f, u|_{\partial S_{R_1}} = g \quad \left(\text{or } u|_{\partial S_R} = f, \quad u_\nu|_{\partial S_{R_1}} = g\right), \tag{2.13}$$

where $u_\nu = \frac{\partial u}{\partial \nu}$ is the exterior normal derivatives on ∂S_{R_1}. On the exterior elliptic boundary ∂S_R, suppose that there exist the approximations of Fourier expansions (see [18]),

$$u \approx a_0 + \sum_{k=1}^{M} \{a_k \cos k\theta + b_k \sin k\theta\} \quad \text{on} \quad \partial S_R, \tag{2.14}$$

$$\frac{\partial u}{\partial \nu} \approx \frac{1}{\sigma_0 \tau_0(\theta)} \left\{ p_0 + \sum_{k=1}^{M} \{p_k \cos k\theta + q_k \sin k\theta\} \right\} \quad \text{on} \quad \partial S_R, \tag{2.15}$$

where a_k, b_k, p_k and q_k are coefficients, and $\tau_0(\theta) = \sqrt{\sinh^2 R + \sin^2 \theta}$. On the interior elliptic boundary ∂S_{R_1}, similarly

[1] For the local ellipse in $\overline{XOY}$, the direction $\overline{X}$ of its long semi-axis can be always chosen with angle $\Theta \in [0, \pi)$.