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Dynamic analysis of Euler–Bernoulli beams by the time-dependent boundary element method formulation

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ABSTRACT

This paper deals with the solution of the Euler–Bernoulli equation for dynamic bending of beams by the time-dependent Boundary Element Method formulation. Initially, an overview of the Euler–Bernoulli beam theory is presented. Next, the time-dependent fundamental solution is introduced and some of its properties are discussed. In the sequence, the integral formulation, obtained through a weighted residuals technique, is presented. Three different numerical implementations are proposed. Finally, the numerical results are compared with the available analytical solutions.

1. Introduction

In the context of the classical theory of beams, or the Euler–Bernoulli beam theory, the reader can find, for instance, the works by Providakis and Beskos [1], Schanz [2], and de Langre et al. [3].

The first of these works is concerned with free and forced flexural vibrations, in which the forced vibration problem is treated with the aid of the Laplace transform and the response is obtained by a numerical inversion of the transformed solution.

In the second work a rather interesting method, called Convolution Quadrature Method, proposed by Lubich [4,5] to perform the convolution in the time-dependent integral equation, is employed. The main characteristics of this method, which can be applied to problems where time-domain fundamental solutions are not available, are (i) the use of fundamental solutions in the Laplace domain and (ii) the numerical approximation of the time convolution integrals, presented in the time-dependent Boundary Element Method (TD-BEM) equations, by a quadrature formula based on a linear method of multiple steps, which provides direct solution in the time-domain.

The third work presents a TD-BEM formulation, which is very interesting from the mathematical point of view. The fundamental solution for this formulation can be found in Graff [6] and Campbell and Foster [7]. One of the conclusions found in reference [3] is transcribed here "It is nevertheless clear that the Laplace transform domain method is the most adequate for the solution of linear problems of flexural vibrations of beams".

This conclusion sounded as a challenge to the authors: if others TD-BEM formulations were successfully developed for other types of problems, e.g., Wrobel [8], Mansur [9], why the same would not be possible for the dynamic analysis of Euler–Bernoulli beams? Bearing this in mind, this paper is concerned with the solution of the Euler–Bernoulli equation for dynamic bending of beams by the TD-BEM formulation. In other words, the motivation was to develop a TD-BEM formulation capable of providing accurate results and, consequently, of encouraging further developments. These include beams over elastic basis, continuous beams, and Timoshenko beams.

Initially an overview of the Euler–Bernoulli beam theory is presented together with the analytical solutions used for comparison with the numerical values provided by the proposed TD-BEM formulation. The time-dependent fundamental solution is then introduced and some of its properties are discussed. In the sequence, the basic TD-BEM equation, obtained by following a weighted residuals approach is presented. Note that the problem is one-dimensional; consequently, the boundary is constituted only by the extreme nodes of the beam. As the domain is the length of the beam, say L, one has $0 \le x \le L$. As the differential equation that governs the problem is non-homogeneous, due to the presence of the loading term, a double integral in space and time, containing this term, appears in the BEM equations. The others integrals are time integrals, evaluated from $t = t_0$ until the

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present time. The beams considered are named according to their boundary conditions as pinned–pinned (PP), clamped–pinned (CP), clamped–clamped (CC), and clamped–free (CF). Four types of loading are considered: the first one is assumed to be linearly distributed along the domain and to act continuously in time (DD); the second type consists of a concentrated load acting continuously in time (CD); the third and the fourth types are short duration loadings; the former is distributed along the domain (DSD) and the later is concentrated (CSD).

Good agreement is observed between the numerical and analytical results, which are presented before the final conclusions. The results for the bending moments and shear forces at internal points are also presented, as these quantities play a fundamental role in the dimensioning of concrete and steel beams.

For problems with non-homogeneous initial conditions, not taken into account here, all the domain or, at least, part of it, must be discretized with cells, as a domain integral containing the non-null initial displacements and rotations appear in the BEM integral equations. Such a discretization can be carried out by employing internal cells, treated as isogeometric boundary elements (see, for instance, [10]).

An interesting extension of the present work could be that based on the assumption of functionally graded materials (see, for instance, [11–13]). In this case, the integral equations would display a domain integral containing the longitudinal elasticity modulus. The same would occur under the assumption of variable transverse section. In general, for variable elasticity modulus and/or moment of inertia, domain integrals appear in the formulation.

2. Euler–Bernoulli beam theory and BEM formulation

Dynamic bending problems of uniform slender beams are governed by the classical Euler–Bernoulli equation

$$EI\frac{\partial^4 u}{\partial x^4}+\rho A\frac{\partial^2 u}{\partial t^2}=f(x,t), \tag{1}$$

where $u(x,t)$ is the vertical deflection of the beam, $f(x,t)$ is the applied external loading, EI is the flexural rigidity of the beam, ρ is the density of the material, A is the cross-sectional area, L is the length of the beam, x is the spatial variable, and t denotes time. Fig. 1 shows an schematic layout of the beam under consideration.

Dividing Eq. (1) by ρA and defining $c=\sqrt{(EI)/(\rho A)}$ one gets the expression

$$c^2\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2 u}{\partial t^2}=\frac{f(x,t)}{\rho A}, \tag{2}$$

which is the governing equation adopted in this work.

From $u(x,t)$ the following quantities are defined:

$$\phi(x,t)=\frac{\partial u}{\partial x},\quad M(x,t)=-EI\frac{\partial^2 u}{\partial x^2},$$
$$Q(x,t)=-EI\frac{\partial^3 u}{\partial x^3},\quad v(x,t)=\frac{\partial u}{\partial t} \tag{3}$$

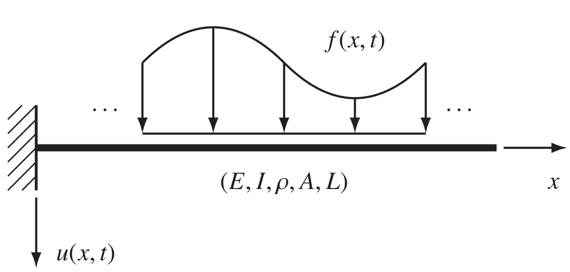

Fig. 1. Schematic layout of the beam under consideration.

called, respectively, rotation, bending moment, shear force, and velocity.

The problem presents four boundary conditions (two at $x=0$ and two at $x=L$) and two initial conditions. The boundary conditions are dependent on the beam type under consideration and are classified as essential boundary conditions, when u or ϕ are prescribed, or natural boundary conditions, when M or Q are prescribed. The initial conditions, conversely, are always of the same type, namely, the values of u and v prescribed at time $t=t_0$.

The analytical solutions of Eq. (2), for the beams considered in this work, can be found, for instance, in Rao [14]. The reader is referred to Sheehan and Debnath [15] for an interesting discussion concerning the theory of the transient Euler–Bernoulli beam on elastic foundation, taking into account the effects of linear damping and axial loading. Another interesting work was presented by Chen et al. [16], which deals with Rayleigh damped Euler–Bernoulli beams subjected to multi-support motion.

The fundamental solution for the problem is the function u^* that satisfies the equation

$$c^2\frac{\partial^4 u^*}{\partial x^4}+\frac{\partial^2 u^*}{\partial t^2}=\delta(x-\xi)\delta(t-\tau), \tag{4}$$

where δ is the Dirac delta function, and corresponds to the deflection of a beam of infinite length measured at a field point x in a time t when an impulsive concentrated load is applied at a source point ξ in a time τ. Such function is given by

$$u^*=\frac{1}{c}\left\{\frac{r}{2}\left[\mathrm{S}\left(\frac{r}{\sqrt{2\pi a}}\right)-\mathrm{C}\left(\frac{r}{\sqrt{2\pi a}}\right)\right]+\frac{\sqrt{a}}{\sqrt{2\pi}}\left[\sin\left(\frac{r^2}{4a}\right)+\cos\left(\frac{r^2}{4a}\right)\right]\right\}, \tag{5}$$

where $r=|x-\xi|$ and $a=c(t-\tau)$.

The solution of Eq. (4) can be found in de Langre et al. [3], which is, to the best of the authors' knowledge, the first BEM formulation which is based on the use of this fundamental solution. However, de Langre et al. [3] did not succeed in obtaining accurate results and concluded that the Laplace transform was the most adequate method for the solution of linear problems of flexural vibrations of beams. The fundamental solution can also be found in Graff [6], Campbell and Foster [7], and Kythe [17].

The functions S and C, called Fresnel integrals, are defined by

$$\mathrm{S}(z)=\int_0^z \sin\left(\frac{\pi}{2}\zeta^2\right)\mathrm{d}\zeta,\quad \text{and}\quad \mathrm{C}(z)=\int_0^z \cos\left(\frac{\pi}{2}\zeta^2\right)\mathrm{d}\zeta. \tag{6}$$

Such integrals have no analytical solution and are evaluated numerically using the algorithm presented by Boersma [18].

The function u^* has the following properties:

- reciprocity: $u^*(x,t,\xi,\tau)=u^*(\xi,-\tau,x,-t)$;
- time translation: $u^*(x,t,\xi,\tau)=u^*(x,t+\Delta t,\xi,\tau+\Delta t)$.

Regarding the characteristics of u^*, it is worth mentioning that u^* does *not* present singularity when $r=0$, though this kind of singularity always occurs in the fundamental solutions of 2D and 3D problems. In a similar fashion, the fundamental solutions for Timoshenko beams listed in Carrer et al. [19] are also *not* singular when $r=0$.

From Eq. (5) one gets

$$\phi^*=+\frac{1}{c}\left\{\frac{1}{2}\left[\mathrm{S}\left(\frac{r}{\sqrt{2\pi a}}\right)-\mathrm{C}\left(\frac{r}{\sqrt{2\pi a}}\right)\right]\right\}\left(\frac{\partial r}{\partial x}\right), \tag{7}$$

$$M^*=-\frac{EI}{c}\left\{\frac{1}{2\sqrt{2\pi a}}\left[\sin\left(\frac{r^2}{4a}\right)-\cos\left(\frac{r^2}{4a}\right)\right]\right\}\left(\frac{\partial r}{\partial x}\right)^2, \tag{8}$$