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An extended exponential transformation for evaluating nearly singular integrals in general anisotropic boundary element method

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ABSTRACT

The exponential transformation is an efficient technique for the accurate numerical evaluation of nearly singular integrals which arise in the boundary element method (BEM). It was shown that this transformation could improve the accuracy of evaluating such integrals by several orders of magnitude. Here, this transformation is extended in a more flexible fashion to allow the evaluation of nearly singular integrals which arise in general anisotropic BEM formulation, with a high degree of accuracy. A major advantage of the new method is its ease of implementation and applicability to a wide class of integrals. Three benchmark test integrals, ranging from nearly weakly, nearly strongly and nearly hyper-strongly singular integrals, are well studied to demonstrate the accuracy and efficiency of the proposed method.

1. Introduction

An important consideration when implementing the boundary element method (BEM) is the accurate and efficient evaluation of nearly singular boundary element integrals [1–4]. These integrals are *nearly* singular when the distance between the calculation point and the element of integration becomes very small and, in this situation, the traditional Gaussian quadrature cannot be used due to a lack of efficiency and accuracy. Such problems are of particular interest in many engineering applications of the BEM, such as the study of thin structures [5,6], contact problems [7], sensitivity problems [8] and displacement around open crack tips [9,10]. These applications involve complex geometries and, in addition, the numerical method used must be able to cope with the fact that the integrand develops a sharp peak as the calculation point moves closer to the element.

Tremendous effort has been devoted to deriving convenient integral forms or sophisticated computational techniques to deal with such problems. The methods developed so far include, but are not limited to, the method of the element subdivision [11–13], analytical or semi-analytical methods [14–17], and various coordinate transformations [18–25]. Impressive results obtained from these techniques have been demonstrated on various examples. Despite these great achievements, it is worth noting that almost all the existing methods share the feature of evaluating nearly singular integrals that only exist in the isotropic BEM formulations. To date, very few studies for such problems arising in anisotropic problems have been reported in the BEM community [26].

This paper focuses on a recent published technique, called the exponential transformation [27]. The key idea of this method is to use an exponential function to remove or smooth out the near singularities of the integrands before conventional Gaussian quadrature is applied. It was shown that this transformation is easy to implement and could improve the accuracy of evaluating nearly singular integrals by several orders of magnitude, compared with conventional Gaussian quadrature. However, as mentioned above, the method was tailored to isotropic problems and cannot be used to general anisotropic problems with it's current form. This paper presents an extension of this method to the evaluation of nearly singular integrals arising in general anisotropic BEM formulations. The new method proposed here is applicable to high-order geometry elements and inherits the merits of the original one of being high accurate, mathematically simple and easy-to-program.

The outline of the rest of the paper is as follows. Section 2 describes the nearly singular integrals which arise in the general anisotropic BEM formulations. The exponential transformation and its numerical implementation are introduced in Sections 3 and 4. Three benchmark examples that are commonly encountered in the applications of the BEM are examined in Section 5. Finally, the conclusions and remarks are provided in Section 6.

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2. The nearly singular integrals in the BEM

Consider a 2D anisotropic medium in an open bounded domain Ω, and assume that Ω is bounded by a surface Γ which may consist of several segments, each being sufficiently smooth in the sense of Liapunov. We also assume that the boundary consists of two parts, $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1, \Gamma_2 \neq \varnothing$ and $\Gamma_1 \cap \Gamma_2 = \varnothing$. In this study, we refer to anisotropic steady heat conduction applications in the absence of inner heat sources. Hence the function $u(\boldsymbol{x})$, which denotes the temperature distribution in Ω, satisfies the equation

$$k_{ij}\frac{\partial^2 u(\boldsymbol{x})}{\partial x_i \partial x_j} = 0, \qquad \boldsymbol{x} \in \Omega, \qquad (i,j=1,2), \tag{1}$$

subject to the boundary conditions

$$u(\boldsymbol{x}) = \overline{u}(\boldsymbol{x}) \quad \text{for} \quad \boldsymbol{x} \in \Gamma_1, \tag{2}$$

$$q(\boldsymbol{x}) = -(K\nabla u(\boldsymbol{x}))\cdot \boldsymbol{n}(\boldsymbol{x}) = -k_{ij}\frac{\partial u(\boldsymbol{x})}{\partial x_j}n_i(\boldsymbol{x}) = \overline{q}(\boldsymbol{x}) \quad \text{for} \quad \boldsymbol{x} \in \Gamma_2, \tag{3}$$

where k_{ij}, $i,j=1,2$ are the thermal conductivity coefficients. $(K_{ij})_{i,j=1,2}$ is assumed to be symmetric and positive-definite so that the partial differential Eq. (1) is elliptic. In addition $\boldsymbol{n}$ denotes the outward normal, the barred quantities $\overline{u}(\boldsymbol{x})$ and $\overline{q}(\boldsymbol{x})$ indicate the measured values of temperature and flux along the boundary. $K = [k_{ij}]$ is the thermal conductivity tensor. The customary standard Cartesian notation for summation over repeated subscripts is employed in this paper. From thermodynamic considerations and Onsager's reciprocity relation, the conductivity coefficients k_{ij} must satisfy

$$k_{11}k_{22} - k_{12}^2 > 0. \tag{4}$$

In the boundary element method, the solution of Eq. (1) can be expressed by the following integral representation

$$C(\boldsymbol{y})u(\boldsymbol{y}) = \int_\Gamma u^*(\boldsymbol{x},\boldsymbol{y})q(\boldsymbol{x})d\Gamma(\boldsymbol{x}) - \int_\Gamma \frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_x}u(\boldsymbol{x})d\Gamma(\boldsymbol{x}), \tag{5}$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are the source and calculation points, respectively. $C(\boldsymbol{y})$ represents a coefficient according to the location of the calculation point and the geometry of the boundary. $u^*(\boldsymbol{x},\boldsymbol{y})$ stands for the fundamental solutions of anisotropic potential problems expressed as follows

$$u^*(\boldsymbol{x},\boldsymbol{y}) = -\frac{1}{2\pi\sqrt{|k_{ij}|}}\ln r(\boldsymbol{x},\boldsymbol{y}), \tag{6}$$

where $|k_{ij}|$ denotes the determinant of k_{ij}, $r(\boldsymbol{x},\boldsymbol{y})$ is the distance between the source and calculation points. Noting that

$$\begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}^{-1} = \frac{1}{|k_{ij}|}\begin{pmatrix} k_{22} & -k_{12} \\ -k_{12} & k_{11} \end{pmatrix}, \tag{7}$$

the distance function $r(\boldsymbol{x},\boldsymbol{y})$ can be expressed as

$$r(\boldsymbol{x},\boldsymbol{y}) = \sqrt{t_{ij}(x_i-y_i)(x_j-y_j)}, \quad (i,j=1,2). \tag{8}$$

An implementation of the BEM requires the accurate evaluation of the following integrals

$$I_1 = \int_{-1}^{1} f(\xi)\log\, r(\xi)d\xi, \tag{9}$$

$$I_2 = \int_{-1}^{1} f(\xi)\frac{1}{r^{2\alpha}(\xi)}d\xi, \qquad \alpha > 0, \tag{10}$$

where ξ is the local intrinsic coordinate, the function $f(\xi)$ denotes a low-order polynomial which may consist of the Jacobian of the transformation from some arbitrarily curved element Γ to line interval $[-1,1]$, shape functions used to interpolate the physical solution and/or the term which arises from taking the derivative of the boundary element kernel.

When the calculation point is far from the boundary element under consideration, a straightforward application of Gaussian quadrature suffices to evaluate such integrals. When the calculation point is on the integral element, the integrand becomes singular and many direct and indirect algorithms have been developed and used successfully [28,29].

A class of integrals which lies between these two extremes is that of the *nearly singular integrals*. Here, the calculation point is close to, but not on, the element and the integrals, theoretically, are regular since the values of their integrands remain finite at all points. However, instead of remaining flat, the magnitude of the integrand may be quite large as the calculation point approaching towards the integral element. The evaluation of such integrals faces considerable difficulties because neither the conventional Gaussian integration nor the methods designed for singular integrals are applicable here.

For nearly singular integrals, when the geometry is approximated using linear element, the distance function r^2 can be expressed as $r^2(\xi) = (\xi - a)^2 + b^2$, where the parameters a ($a \in [-1,1]$) and $b(b>0)$ represent the position of the nearly singular point and the shortest distance to the element [20,26], respectively, as shown in Fig. 1. In this case, both integrals I_1 and I_2, as shown in Table 1, can be evaluated analytically using various exact integration methods [14,16,30–32]. Detailed discussion of such issues is outside the scope of this paper. When the geometry is approximated using high-order geometry element (usually of second order) [26,33], the distance function r^2 has the form of $r^2(\xi) = (\xi - a)^2 g(\xi) + b^2$, where $g(\xi) > 0$ is a well-behaved function and the parameters a and b defined above still remain the same. In this case, the Jacobian of the integral is not a constant but a non-rational function, making exact integration difficult. Nevertheless, other numerical techniques should be developed for their evaluation. This is one of the purposes of this paper.

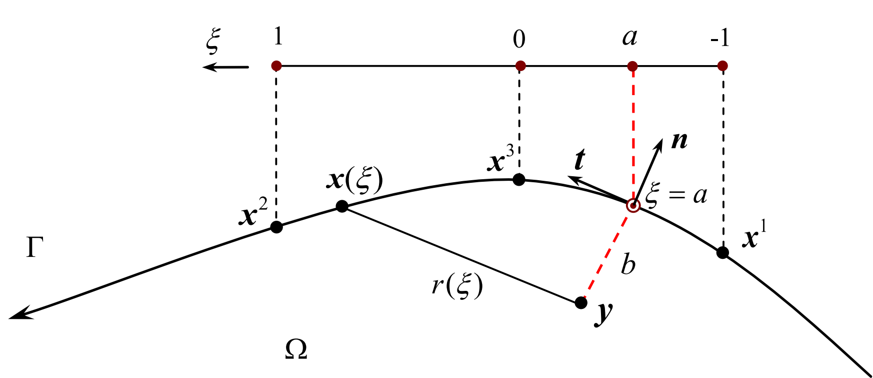

Fig. 1. Geometry of a parabolic boundary element.