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Application of the method of fundamental solutions to 2D and 3D Signorini problems

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ABSTRACT

This paper presents an application of the method of fundamental solutions (MFS) for the numerical solution of 2D and 3D Signorini problems. In our application, by using a projection technique to tackle the nonlinear Signorini boundary inequality conditions, the original Signorini problem is transformed into a sequence of linear elliptic boundary value problems and then solved by the MFS. Convergence and efficiency of the present MFS is proved theoretically and verified numerically.

1. Introduction

Signorini problems come up in the modeling of many realistic science and engineering applications such as the shallow dam problem [1–4], the electropaint process [3–7], the unilateral contact problem [8,9], the free boundary problem [10,11], and so on. Besides, in the theory of variational inequalities [12,13], a broad class of problems arising in industry, social, economics, finance, pure and applied sciences give rise to Signorini problems. The numerical solution of Signorini problems has been frequently dominated by classical numerical methods such as the finite difference method and the finite element method (FEM) [1,6,8]. These methods require domain meshing which is often arduous, computationally expensive, and fraught with pitfalls.

In Signorini problems, the boundary potential and its normal derivative alternate on the Signorini boundary in conjunction with certain nonlinear boundary inequality conditions. To obtain the solution in the domain, we need first to determine the number and position where the change from one type of boundary condition to the other occurs. Therefore, the primary focus in solving Signorini problems is on the Signorini boundary of the domain and thus, the boundary element method (BEM) is particularly suitable for the approximate solution of such problems. Some applications of the BEM for Signorini problems can be found in Refs. [2,3,9,11,14–17].

The BEM reduces the computational dimensions of the original problem by one and gives a simple discretization of infinite domain problems [18], but it involves the generation of elements on the boundary surface and the computation of some complex singular integrals on boundary elements. In some cases, these processes can also be very difficult and computationally expensive. To alleviate the meshing-related issues, some boundary type meshless methods, such as the boundary node method (BNM) [19], the boundary point interpolation method (BPIM) [20], the hybrid BNM [21–25], the Galerkin BNM [26,27], the dual BNM [28,29] and the boundary element-free method (BEFM) [30–32], have been developed by introducing meshless shape functions into boundary integral equations. In recent years, the BNM [33], the BPIM [34] and the BEFM [35] have been extended to 2D Signorini problems. However, these boundary type meshless methods still involve the computation of boundary integrals.

The method of fundamental solutions (MFS) [36,37] is a boundary type meshless method, in which the solution is approximated as a linear combination of fundamental solutions. The MFS eliminates the issue of computing integrals required in the BEM and the rest boundary type methods aforementioned. Being mathematically simple, integration-free, easy-to-program, truly meshless and the extensible to multidimensional problems make the MFS very attractive in solving boundary value problems. In 1998, Poullikkas et al. pioneered the application of the MFS to 2D Signorini problems [4]. In 2001, they also applied the MFS to a special 3D Signorini problem, known as the electropainting problem [5]. In their works, the original problem is reformulated as a nonlinear least-squares problem with nonlinear inequality constraints, and then solved by a special

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least-squares minimization routine to accommodate constraints. However, the success of their application depends severely on the quality of the optimization algorithm used to handle the nonlinear constrained problem.

In this paper, a new application of the MFS is developed for boundary-only analysis of Signorini problems. The numerical formulae are valid for 2D and 3D Signorini problems and also valid for both interior and exterior problems simultaneously. In our application, the nonlinear boundary inequality conditions are incorporated naturally into an iterative scheme by using a projection operator [12,13]. Then, the original Signorini problem is transformed into a sequence of well-posed linear boundary value problems. Finally, the MFS is used iteratively for solving these linear problems. The application of the MFS in this paper, differs from that in Refs. [4,5], involves only linear boundary value problems and linear system of algebraic equations, and thus avoids the design of a quite sophisticated and time-consuming optimization algorithm. As a result, the present application is expected to have higher computational speed and efficiency. Convergence and efficiency of the present MFS is also proved theoretically and verified numerically in detail.

The following discussions begin with a detailed numerical implementation of the MFS for Signorini problems in Section 2. Then, Section 3 provides the associated iterative algorithm and convergence analysis. Finally, numerical examples and conclusions are given in Sections 4 and 5, respectively.

2. The MFS for Signorini problems

Let Ω be a d-dimensional domain in $\mathbb{R}^d$, where $d=2$ or 3 denotes the spatial dimension. A generic point in $\mathbb{R}^d$ is denoted as $\mathbf{x}=(x_1,x_2,\ldots,x_d)^{\mathrm{T}}$ or $\mathbf{y}=(y_1,y_2,\ldots,y_d)^{\mathrm{T}}$. Let Γ_D, Γ_N and $\Gamma_S\neq\varnothing$ be three disjointed parts constituting the boundary Γ of Ω.

Consider the following Signorini problem:

$$\Delta u(\mathbf{x})=0,\quad \mathbf{x}\in\Omega \tag{1}$$

$$u(\mathbf{x})=\phi(\mathbf{x}),\quad \mathbf{x}\in\Gamma_D \tag{2}$$

$$q(\mathbf{x})=\varphi(\mathbf{x}),\quad \mathbf{x}\in\Gamma_N \tag{3}$$

and

$$u(\mathbf{x})=\phi(\mathbf{x}),\quad q(\mathbf{x})<\varphi(\mathbf{x}),\quad \mathbf{x}\in\Gamma_S \tag{4}$$

or

$$u(\mathbf{x})<\phi(\mathbf{x}),\quad q(\mathbf{x})=\varphi(\mathbf{x}),\quad \mathbf{x}\in\Gamma_S \tag{5}$$

where $u\in H^1(\Omega)$ is the unknown function, $q=\partial u/\partial\mathbf{n}$ is the normal derivative of u, $\mathbf{n}=(n_1,n_2,\ldots,n_d)^{\mathrm{T}}$ is the unit outward normal to Γ, and $\phi\in H^{1/2}(\Gamma_D\cup\Gamma_S)$ and $\varphi\in H^{-1/2}(\Gamma_N\cup\Gamma_S)$ are prescribed boundary functions.

To obtain the solution u in Ω, we need first to determine on which parts of Γ_S boundary conditions (4) apply, and on the remaining parts boundary conditions (5) apply. Then, the solution u and its derivatives in Ω can be solved using a standard numerical method such as the FEM, the BEM and meshless methods. As a result, the primary focus in solving Signorini problem (1)–(5) is on the Signorini boundary of the domain and thus, the inequality Signorini boundary conditions (4) and (5) need to be tackled efficiently. For doing this, it can be verified firstly that conditions (4) and (5) are equivalent to the following boundary conditions:

$$u(\mathbf{x})\le\phi(\mathbf{x}),\quad q(\mathbf{x})\le\varphi(\mathbf{x}),\quad (u(\mathbf{x})-\phi(\mathbf{x}))(q(\mathbf{x})-\varphi(\mathbf{x}))=0,\quad \mathbf{x}\in\Gamma_S \tag{6}$$

Then, let [12,13]

$$\mathcal{P}a:=\min(a,0),\quad a\in\mathbb{R} \tag{7}$$

where $\mathcal{P}$: $\mathbb{R}\to\mathbb{R}^-\cup\{0\}$ is a projection operator. And let

$$q(\mathbf{x})-\varphi(\mathbf{x})=\mathcal{P}[(q(\mathbf{x})-\varphi(\mathbf{x}))-\alpha(u(\mathbf{x})-\phi(\mathbf{x}))],\quad \mathbf{x}\in\Gamma_S \tag{8}$$

where α is an arbitrary positive constant.

In light of Eq. (7), from Eq. (8) we have $q\le\varphi$. If $q=\varphi$, then $(q-\varphi)-\alpha(u-\phi)\ge 0$, and thus $u\le\phi$. Otherwise, if $q<\varphi$, then $(q-\varphi)-\alpha(u-\phi)<0$, and thus recalling again Eq. (8) leads to $u=\phi$. As a result, from Eq. (8) we can deduce the Signorini boundary conditions (4) and (5), and thus deduce Eq. (6). On the other hand, from Eq. (6) we can deduce Eq. (8) immediately. Summarizing, we have shown that Eq. (6) is equivalent to Eq. (8).

In view of Eq. (8), an implicit iterative scheme can be defined for numerical computation as

$$q^{(k+1)}(\mathbf{x})=\varphi(\mathbf{x})+\mathcal{P}\left[\left(q^{(k)}(\mathbf{x})-\varphi(\mathbf{x})\right)-\alpha\left(u^{(k+1)}(\mathbf{x})-\phi(\mathbf{x})\right)\right],\quad k=0,1,2,\ldots\ \mathbf{x}\in\Gamma_S \tag{9}$$

where the superscript (k) denotes the value at the kth iteration.

Since the projection operator $\mathcal{P}$ is also nonlinear, Eq. (9) cannot be implemented directly. In this study, this operator is checked point-wise. Let $\{\mathbf{x}_i\}_{i=1}^N\subset\Gamma$ be a set of N boundary nodes which are necessary for the implementation of the MFS. After the kth iteration, both $u^{(k)}(\mathbf{x}_i)$ and $q^{(k)}(\mathbf{x}_i)$ are known for all $\mathbf{x}_i\in\Gamma_S$. Thus, if on $\Gamma_{SN}\subset\Gamma_S$ inequality

$$\left(q^{(k)}(\mathbf{x}_i)-\varphi(\mathbf{x}_i)\right)-\alpha\left(u^{(k)}(\mathbf{x}_i)-\phi(\mathbf{x}_i)\right)>0 \tag{10}$$

is true for all $\mathbf{x}_i\in\Gamma_{SN}$, and for $\mathbf{x}_i\in\Gamma_{SR}:=\Gamma_S\backslash\Gamma_{SN}$ this inequality is false, then from Eq. (9) we use

$$q^{(k+1)}(\mathbf{x}_i)=\varphi(\mathbf{x}_i),\quad \mathbf{x}_i\in\Gamma_{SN} \tag{11}$$

and

$$q^{(k+1)}(\mathbf{x}_i)=q^{(k)}(\mathbf{x}_i)-\alpha\left(u^{(k+1)}(\mathbf{x}_i)-\phi(\mathbf{x})_i\right),\quad \mathbf{x}_i\in\Gamma_{SR} \tag{12}$$

for the $(k+1)$th iteration. It should be stressed that the MFS is incidental to check and update the inequality given by Eq. (10).

According to Eqs. (11) and (12), the original Signorini problem (1)–(5) is reduced to the following linear problem:

$$\Delta u^{(k+1)}(\mathbf{x})=0,\quad \mathbf{x}\in\Omega \tag{13}$$

$$u^{(k+1)}(\mathbf{x})=\phi(\mathbf{x}),\quad \mathbf{x}\in\Gamma_D \tag{14}$$

$$q^{(k+1)}(\mathbf{x})=\varphi(\mathbf{x}),\quad \mathbf{x}\in\Gamma_N \tag{15}$$

$$q^{(k+1)}(\mathbf{x}_i)=\varphi(\mathbf{x}_i),\quad \mathbf{x}_i\in\Gamma_{SN} \tag{16}$$

$$\alpha u^{(k+1)}(\mathbf{x}_i)+q^{(k+1)}(\mathbf{x}_i)=q^{(k)}(\mathbf{x}_i)+\alpha\phi(\mathbf{x}_i),\quad \mathbf{x}_i\in\Gamma_{SR} \tag{17}$$

where $k=0,1,2,\ldots$

The boundary value problem (13)–(17) is now solved using the MFS.

With the help of the MFS, the solution of this problem can be approximated as

$$u^{(k+1)}(\mathbf{x})\approx u_N^{(k+1)}(\mathbf{x})=\sum_{j=1}^{N}a_j^{(k+1)}U\left(\mathbf{x},\mathbf{y}_j\right),\quad \mathbf{x}\in\overline{\Omega}=\Omega\cup\Gamma,\ k=0,1,2,\ldots \tag{18}$$

where $a_j^{(k+1)}$ is the jth unknown coefficient at the $(k+1)$th iteration, $\mathbf{y}_j$ is the source point placed on a fictitious boundary Γ', N is the number of source points, and

$$U\left(\mathbf{x},\mathbf{y}_j\right)=\frac{1}{2(d-1)\pi}\cdot\begin{cases}-\ln\left|\mathbf{x}-\mathbf{y}_j\right|, & d=2\\ \dfrac{1}{\left|\mathbf{x}-\mathbf{y}_j\right|}, & d=3\end{cases}$$

is the fundamental solution of Laplace's equation.