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The modified dual reciprocity boundary elements method and its application for solving stochastic partial differential equations

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ABSTRACT

This paper proposes a numerical method based on the dual reciprocity boundary elements method (DRBEM) to solve the stochastic partial differential equations (SPDEs). The concept of dual reciprocity method is used to convert the domain integral to the boundary. The conventional DRBEM starts with approximation of the source term of the original PDEs with radial basis functions (RBFs). Due to the fact that the nonhomogeneous term of SPDEs considered in this paper involves Wiener process, the traditional DRBEM cannot be applied. So a modification of it is suggested that has some advantages in comparison with the traditional DRBEM and can be developed for solving the SPDEs.

The time evolution is discretized by using the finite difference method, while the modified DRBEM is proposed for spatial variations of field variables. The noise term is approximated at the collocation points at each time step. We employ the generalized inverse multiquadrics (GIMQ) RBFs to approximate functions in the presented technique. To confirm the accuracy of the new approach, several examples are employed and simulation results are reported. Also the convergence of the new technique is studied numerically.

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1. Introduction

Stochastic partial differential equations (SPDEs) provide a quantitative description for mathematical models in areas such as physics, engineering, biology, geography and finance. Actually many phenomena, both in nature and engineering which are described by deterministic partial differential equations (PDEs), can be more fully modeled by systems of SPDEs, for instance see [1]. For example the authors of [52] suggest the use of a model based on the spatially inhomogeneous, nonlinear Smoluchowski equations with random initial distribution to describe the annihilation of spatially separate electrons and holes in a disordered semiconductor. Also the Lagrangian stochastic models are suggested for simulating the transport of particles in turbulent flows in [53]. For another example stochastic cable equation arises in neurophysiology. This particular example cames up in connection with a study of neurons. These nerve cells are the building blocks of the nervous system, and operate by a mixture of chemical, biological and electrical properties, for more details see [55]. The elliptic SPDEs occur for example in random vibrations, seismic activity, oil reservoir management and composite materials, see for example [6,28,29] and the references therein.

The initial and boundary value problems of SPDEs have been studied theoretically, for example see [19,22,55]. However, it is difficult to obtain the analytical solutions of SPDEs. So, the numerical solution of SPDEs becomes a fast growing research area. The finite difference and finite element methods [2,13], the Wiener chaos expansion [39], the stochastic spectral collocation method [45], the Itô Taylor expansions method [40] are discussed for approximating SPDEs arising in engineering and science. Another numerical technique that has been applied for solving SPDEs is based on the meshless methods. Meshless methods have been applied for the numerical solution of time-dependent SPDEs [20,59] and for time-independent SPDEs [30]. Also meshless methods applied for the numerical solution of nonlinear SPDEs [60]. In addition, references [11,12,35] provide useful works in the numerical solution of stochastic differential equations.

The boundary elements method (BEM) has become powerful tool for the numerical study of some engineering problems modeled by deterministic [3–5,8,10,23–26,38] and stochastic [16,32,41,42,49] PDEs. The main idea in this method is to convert the original PDE to an equivalent boundary integral equation by using Green's theorem and a fundamental solution of the original equation. Consequently the main advantage in this method over the classical domain methods such as finite element, finite difference and finite volume methods is that only boundary discretization is required due to dimension reduction. The

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BEM requires a fundamental solution to the original differential equation in order to avoid domain integrals in the formulation of the boundary integral equation, which is one of the drawbacks of the BEM. Another drawback is that nonhomogeneous and nonlinear terms are incorporated in the formulation by means of domain integrals. The use of cells to evaluate these domain integrals implies an internal discretization which considerably increases the quantity of data necessary to run a problem. Thus, the method loses the attraction of its boundary-only character in relation to the other domain decomposition methods. One of the most efficient methods to eliminate the domain integrals is the dual reciprocity BEM (DRBEM) [17,18,24]. The main idea behind this approach is to approximate inhomogeneous term of the considered PDEs by interpolation in terms of some well-known functions $\phi(r)$, named radial basis functions (RBFs) [36]. There exists a large class of interpolating RBFs [46] that can be used by DRBEM. Some well-known RBFs [54] that are employed as approximate function in DRBEM are listed bellow:

- Linear: $1+r$.
- The thin plate spline: $r^{2n}\log(r), \quad n=1,2,\ldots$.
- The Gaussian: $\exp(-cr^2)$.
- The multiquadrics: $\sqrt{r^2+c^2}$.
- The inverse multiquadrics: $\frac{1}{\sqrt{r^2+c^2}}$.
- The generalized inverse multiquadrics: $\frac{1}{(r^2+c^2)^2}$.

where r is the distance between a source point and the field point and c is a constant shape parameter.

1.1. Problem setting

Suppose that $\mathcal{D}\subset\mathbb{R}^2$ is a regular open bounded domain and $\mathcal{H}$ is a separable Hilbert space of function defined on $\mathcal{D}$. The main concern in this paper is solving the following parabolic SPDEs [7,19,20,22,61,57,58]:

$$\begin{cases} du=(\Delta u+f)\,dt+\sigma dW(t) & \text{in } \mathcal{D}, \quad 0<t<T, \\ u(x,0)=u_0\in\mathcal{H}, & x\in\mathcal{D}, \\ u(x,t)=g, & x\in\partial\mathcal{D}, \end{cases} \tag{1}$$

where W is a Wiener process defined on filtered probability space $(\Omega_W,\mathcal{F}_W,\{\mathcal{F}_t\}_{t=0}^{\infty},\mathbb{P}_W)$ with mean zero and spatial covariance function q given by

$$\mathbb{E}(W(t,x)W(s,y))=\min\{t,s\}q(x,y), \quad x,y\in\mathcal{D}, \quad t,s>0.$$

In addition, Δ is the Laplace operator, $\sigma>0$ and the functions f and g are given such that the problem (1) has a unique solution [19]. The existence, uniqueness and properties of the solutions of such equations have been well studied, for instance see Da Prato and Zabczyk [22], Walsh [55], etc. Allen et al. [2], Davie and Gaines [21], Du and Zhang [27], Gyöngy [31], Hausenblas [33,34], Kloeden and Shott [47], Lord and Rougemont [48], Cialenco et al. [20], Ye [59,61] and Yan [57,58] are some works on the numerical solution of Eq. (1).

In this paper we employ a modification of the conventional DRBEM for the numerical solution of Eq. (1). The modified DRBEM is a BEM-like meshless method. Some other BEM-like meshless method such as method of fundamental solution [15], boundary knot method [14,37] and singular boundary method [56] have been widely used in the literatures for solving engineering problems. Also a mesh-free stochastic boundary method based on randomized versions of the method of fundamental solutions was presented in [51]. The organization of the current paper is as follows:

In Section 2 we mention some drawback of the traditional DRBEM and explain why the conventional DRBEM cannot be applied for the numerical solution of SPDEs. So in Section 3 a modification of the conventional DRBEM for solving the deterministic PDEs is suggested. The proposed idea can be developed for solving SPDEs easily. So, the stochastic Poisson and stochastic heat equations are solved via the method developed in Section 4. Section 5 contains numerical experiments which show the hight performance of the presented method.

2. Motivation

Suppose $\mathcal{D}\subset\mathbb{R}^2$ is a bounded computational domain with piecewise smooth boundary Γ. Consider the following Poisson equation:

$$\begin{cases} \Delta u=b & \text{in } \mathcal{D}\subset\mathbb{R}^2, \\ u=\overline{u} & \text{on } \Gamma_u, \\ \dfrac{\partial u}{\partial n}=\overline{u}_n & \text{on } \Gamma_{u_n}, \end{cases} \tag{2}$$

when $\overline{u}$ and $\overline{u}_n$ are known values of potential and flux, respectively, b is a known function of position, n is the outward normal vector over the boundary $\Gamma=\Gamma_u\cup\Gamma_{u_n}$.

Suppose $G^i=G^i(x,y)$ be the fundamental solution of Laplace equation based on the source point (x_i,y_i), i.e.

$$\Delta G^i=\delta(x-x_i,y-y_i), \tag{3}$$

where $\delta(x,y)$ is the Dirac delta function, and (x_i,y_i) and (x,y) are source and field points, respectively. It is well-known that the fundamental solution for 2D Laplace equation based on the source point (x_i,y_i) is reported as [44]

$$G^i(x,y)=\frac{1}{2\pi}\ln(r), \tag{4}$$

where r is the distance between the field point (x,y) and the source point (x_i,y_i), i.e. $r=\sqrt{(x-x_i)^2+(y-y_i)^2}$. Multiplying the first equation of Eqs. (2) with the weighting function G^i and applying Green's second theorem lead to the integral formulation

$$c_iu_i+\int_\Gamma\left(G^i\frac{\partial u}{\partial n}-u\frac{\partial G^i}{\partial n}\right)d\Gamma=\int_{\mathcal{D}}bG^i d\mathcal{D}, \tag{5}$$

where $c_i=\frac{1}{2\pi}\alpha_0$ such that α_0 is the internal angle of the boundary at the source point [43]. It is well-known that $\alpha_0=2\pi$ when the collocation point is inside Ω and $\alpha_0=\pi$ when it is located on the smooth parts of Γ [9,43].

The domain integral on the right hand side of Eq. (5) still remains in the BEM, this integral can be evaluated by dividing the domain into cells [46]. The motivation behind DRBEM is to avoid this procedure by transforming the domain integral to an equivalent boundary integral. This can be achieved by approximating the function b in terms of RBFs at some chosen number of boundary (N) and internal (L) nodes in the domain. So the function b can be expressed as

$$b=\sum_{j=1}^{N+L}\alpha_j\phi_j, \tag{6}$$

where ϕ_j represents the interpolation function, ϕ, from a field node to source node, i.e.

$$\phi_j:=\phi(\|x-x_j\|), \quad x,x_j\in\mathcal{D},$$

where $\|x-x_j\|$ denotes the distance between x and x_j and $\alpha_j, j=1,\ldots,N+L$, are the corresponding interpolating coefficients. The essential feature in DRBEM is to express ϕ_j, which is a function of r_j, as a Laplacian of another function ψ_j. Thus ψ_j is chosen as the

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