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The numerical solution of Cahn–Hilliard (CH) equation in one, two and three-dimensions via globally radial basis functions (GRBFs) and RBFs-differential quadrature (RBFs-DQ) methods

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ABSTRACT

The present paper is devoted to the numerical solution of the Cahn–Hilliard (CH) equation in one, two and three-dimensions. We will apply two different meshless methods based on radial basis functions (RBFs). The first method is globally radial basis functions (GRBFs) and the second method is based on radial basis functions differential quadrature (RBFs-DQ) idea. In RBFs-DQ, the derivative value of function with respect to a point is directly approximated by a linear combination of all functional values in the global domain. The main aim of this method is the determination of weight coefficients. GRBFs replace the function approximation into the partial differential equation directly. Also, the coefficients matrix which arises from GRBFs is very ill-conditioned. The use of RBFs-DQ leads to the improvement of the ill-conditioning of interpolation matrix RBFs. The boundary conditions of the mentioned problem are Neumann. Thus, we use DQ method directly on the boundary conditions, which easily implements RBFs-DQ on the irregular points and regions. Here, we concentrate on Multiquadrics (**MQ**) as a radial function for approximating the solution of the mentioned equation. As we know this radial function depends on a constant parameter called shape parameter. The RBFs-DQ can be implemented in a parallel environment to reduce the computational time. Moreover, to obtain the error of two techniques with respect to the spatial domain, a predictor–corrector scheme will be applied. Finally, the numerical results show that the proposed methods are appropriate to solve the one, two and three-dimensional Cahn-Hilliard (CH) equations.

1. Introduction

The CH equation is a forth-order nonlinear partial differential equation as follows [53,89]:

$$\frac{\partial u(\mathbf{x},t)}{\partial t} = \nabla\cdot(M\nabla(F'(u(\mathbf{x},t)) - \varepsilon^2\nabla^2 u(\mathbf{x},t))), \quad u(\mathbf{x},t)\in[-1,1],\ \mathbf{x}\in\Omega, \tag{1.1}$$

where $u(\mathbf{x},t)$ is the concentration of a component within a binary mixture, $F'(u(\mathbf{x},t))$ is a free energy function, and ϵ is the gradient interfacial energy coefficient. Also, M is a constant mobility. The split form of the CH equation is [53,89]

$$\begin{cases} \dfrac{\partial u(\mathbf{x},t)}{\partial t} = M\nabla^2\mu(\mathbf{x},t), \\ \mu(\mathbf{x},t) = F'(u(\mathbf{x},t)) - \varepsilon^2\nabla^2 u(\mathbf{x},t), \end{cases} \tag{1.2}$$

where μ is the local chemical potential. The boundary conditions of (1.2) are homogenous Neumann boundary as follows:

$$\nabla u\cdot\mathbf{n} = \nabla\mu\cdot\mathbf{n} = 0, \tag{1.3}$$

where ∇u and $\nabla\mu$ are gradient vectors and $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega$. As is said in [73], the CH equation arises from the

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Ginzburg–Landau free energy

$$E(u) := \int_{\Omega} \left(F(u) + \frac{\varepsilon^2}{2} |\nabla u|^2 \right) d\mathbf{x}, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n (n = 1, 2, 3)$. Note that there are different types of the free energy such as [53,60,89]

$$F(u) = \tfrac{1}{4} u^2 (1-u)^2, \quad F(u) = \tfrac{1}{4} (u^4 - 2u^2), \quad F(u) = \tfrac{1}{4} (1-u^2)^2, \tag{1.5}$$

or logarithmic free energy [73,89] i.e.

$$F(u) = A(u \ln(u) + (1-u)\ln(1-u) + Bu(1-u)). \tag{1.6}$$

As is said in [89], in Eq. (1.6), the constant B is a notion of the interaction between components, i.e. miscibility. Also when B is negative, components are miscible, while when B is positive, the components repel each other. The degree of miscibility can change with external conditions, e.g. temperature, concentration or pressure. Also the CH equation describes the evolution of the system which follows energy minimization while conserving mass [89]. In this paper we use Eq. (1.5) as the free energy.

1.1. The CH equation and its applications

The CH equation was originally proposed by Cahn and Hilliard to model the spinodal decomposition and coarsening phenomena observed in binary alloys [12,13]. Also, the CH equation is used by Anders et al. to describe laser-induced restructuring of thin polymer films [3,4]. As is said in [60], one of the essential concepts of the CH equation is the interface between two phases which said α and β. It has a finite thickness in which the composition c changes gradually. When the binary system approaches near the equilibrium state composed of α phase with $c = c_\alpha^{eq}$ and β phase with $c = c_\beta^{eq} > c_\alpha^{eq}$, the domains where $c(\mathbf{x}, t) = c_\alpha^{eq}$ and $c(\mathbf{x}, t) = c_\beta^{eq}$ correspond to the α and β phases, respectively, whereas the region where $u(\mathbf{x}, t)$ varies gradually from c_α^{eq} to c_β^{eq} represents the interface between the α and β phases (see Fig. 1).

Also for simulating microstructural evolution, using the CH equation makes the avoidance of explicit tracking of the interface [60]. As is mentioned in [60], this concept of a diffuse interface has been adopted to model various physical phenomena involving moving interfaces, in which the order parameter, or phase field $\phi(\mathbf{x}, t)$, instead of the composition field $u(\mathbf{x}, t)$, is introduced to describe the spatial distribution of the entire microstructure of a system. Some applications of the CH equation are pointed in [53] such as the phase separation of binary and ternary liquid mixture [1,70], multi-phase fluid flows [9,57,54,55], Taylor flow in mini/microchannels [38], two-layer flow in channels with sharp topographical features [95], spinodal decomposition with composition-dependent heat conductivities [69], phase decomposition and coarsening in solder balls [2], the thermal-induced phase separation phenomenon [83], the evolution of arbitrary morphologies and complex microstructures such as solidification, solid-state structural phase transformations [14,40,47,65], meta-stable chemical composition modulations in the spinodal region [39], modeling of martensitic phase transformation [66], grain growth [92], pore migration in a temperature gradient [93], image in painting [7,8], and tumor growth [17,88]. Other applications of CH equation also are image processing [27], planet formation [84] and cancer growth [59].

1.2. The literature review

In this subsection we briefly review the numerical methods for solving the CH equation. Various numerical methods have been developed for solving the mentioned equation in recent years. Some of these methods are Galerkin finite element method [29,30], second-order splitting method [31], nonconforming finite element method [32], numerical analysis with a logarithmic free energy [18], unconditionally gradient stable scheme [33], semi-implicit Fourier-spectral method [15,96], stable and conservative finite difference technique [37], conservative multigrid method [56], discontinuous Galerkin method [86], moving mesh method [35], adaptive mesh refinement idea [88], boundary integral method [19], local discontinuous Galerkin method [90], large time-stepping method [43], isogeometric analysis procedure [41], strongly anisotropic CH equation by an adaptive nonlinear multigrid method [87], conservative scheme with contact angle boundary condition [58], conservative scheme with Neumann and Dirichlet boundary conditions in complex

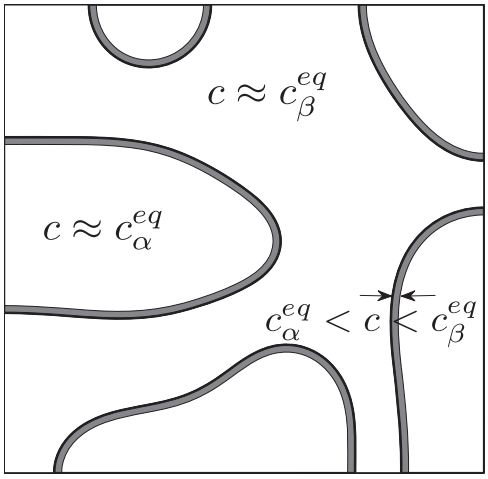

Fig. 1. Two phase microstructure with order parameter u [60].