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FEM/BEM formulation for multi-scale analysis of stretched plates

G.R. Fernandes [*,a], J.J.C. Pituba [a], E.A. de Souza Neto [b]

[a] Civil Engineering Department Campus Catalão – Federal University of Goiás (UFG) CAC, Av. Dr. Lamartine Pinto de Avelar, 1120, Setor Universitário- CEP, 75700-000 Catalão, GO, Brazil

[b] College of Engineering, Swansea University, Singleton Park, Swansea SA2 8PP, United Kingdom

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ABSTRACT

A multi-scale modelling for analysing the stretching problem of plates composed of heterogeneous materials is presented. The BEM (Boundary Element Method) is adopted to model the macro-continuum (represented by the plate) while the equilibrium problem at micro-scale (represented by the Representative Volume Element – RVE) is solved by a FEM (Finite Element Method) formulation that takes into account the Hill–Mandel Principle of Macro-Homogeneity. After solving the equilibrium problem of the RVE, the micro-to-macro transition is made by applying the volume averaging hypothesis of strain and stress tensors. Some numerical examples are then analysed to show that the proposed formulation is a suitable tool for the analysis of stretched of plates composed of heterogeneous materials. To define the microstructure, different RVEs composed of an elasto-plastic matrix with inclusions or voids are considered. Besides, a quadratic rate of asymptotic convergence of the Newton–Raphson scheme has been achieved during the iterative procedure.

1. Introduction

Despite of the materials used in Engineering (metals, polymers, composites, concretes and woods) have different microstructures, at the macroscopic level similar characteristics of mechanical behaviour are observed, as example: elasticity, viscosity, plastic strain, brittle rupture, ductile rupture, etc. Because of these similarities, constitutive models based on continuum mechanics and thermodynamics of solids applied to macroscopic analyses are usually proposed. However, it is important to note that the deformation and rupture processes take place at micro-scale level. Accordingly, many works have been developed in order to analyse the dissipative phenomena in the micro-scale of materials using many techniques and constitutive models [1–5]. In this context, modelling heterogeneous material in different scales is very important to better represent the behaviour of such complex materials [6–10]. In many situations the traditional phenomenological approach for constitutive description does not provide a sufficiently general predictive modelling capability [11–13].

The infinitesimal material neighbourhood of a point at the macro-continuum is represented by the RVE (representative volume element) where the material behaviour is monitored individually (see [14–17]). Therefore the micro-structure is represented by the RVE, whose equilibrium problem is solved after imposing to the RVE the macro strain related to the point at the macro-continuum. After solving the equilibrium problem at micro-scale, the micro-to-macro transition can be made by applying a homogenisation process. Thus the micro-scale passes information to the macro-scale and vice versa. Note that the integrity of a local point of the macro-continuum is reduced if dissipative processes take place in the microstructure. Then, a non-linear formulation is also required to model the macro-continuum. In this context, the most of the works uses the Finite Element Method (FEM) to model the mechanical behaviour of all scales involved in the problem [18–27], although some few formulations using Boundary Element Method (BEM) have been proposed to model the mechanical behaviour into the context of multi or microscale [28–31].

On the other hand, the BEM is a suitable tool to deal with plate problems (see [32–36]), being specially indicated to compute displacements and forces due to stress or strain concentration problems. This can occur when the plate is subjected to loads distributed over small regions or due to a fracture process leading to strain localisation, for example. In future works the authors intend to perform multi-scale analysis of plates composed of brittle materials where the fracture process can be very important. Thus, with this work, we intend to present the BEM as a new alternative method to deal with multi-scale analysis of structures composed of heterogeneous materials.

In the present work, the non-linear BEM formulation for analysing the stretching problem of plates presented in [37] is used to model the macro-continuum. The BEM integral representation for in-plane displacements is obtained from Betti's theorem where the initial force field over the domain is approximated by using the well-known cell subdivision (see [37–39]). The equilibrium problem to be solved at micro-scale is defined by the FEM formulation developed in [14], where the problem consists of finding the field of displacement fluctuation that,

* Corresponding author.

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for each instant t, the RVE equilibrium equation is satisfied. This displacement fluctuation represents how the microscopic strain varies over the RVE, i.e., in the case of having uniform microscopic strain, the displacement fluctuation is null and the macroscopic strain coincides to the microscopic strain. To solve the RVE equilibrium problem, different boundary conditions can be adopted for the RVE, leading to different multi-scale models and therefore to different numerical responses. In the present work the following boundary conditions can be considered in the RVE: (i) linear displacements, (ii) periodic displacement fluctuations and (iii) uniform boundary tractions. Besides, an algorithm is presented to solve the proposed multi-scale modelling. Finally, two numerical examples are presented. In the first one, a plate subjected to uniform normal stress is considered, where a RVE composed of several inclusions (or voids) is adopted. We have performed different analysis where the inclusions have been considered elastic or elasto-plastic and whose material properties have been changed in order to verify how the inclusions can affect the plate mechanical response. Then, considering the RVE with several voids, different boundary conditions are imposed to the RVE. The second example consists of a perforated plate subjected to normal stress where an elastic inclusion is defined at the RVE central region. In this case, we have considered different volume fractions for the inclusion in order to observe how the inclusion can affect the plate mechanical behaviour. This kind of RVE can represent materials as the MMCs (metal matrix composites) where inclusions are added to the material in order to improve its elastic properties, such as: high stiffness, high tensile strength, creep resistance, wear resistance, low density and damping capabilities.

Note that in [40] the authors present a multi-scale model to analyse the simple bending problem, i.e., a BEM formulation for plate bending analysis has been adopted for the macro-continuum. The FEM formulation used in [40] to model the RVE is the same considered in the present work, although the number of RVEs required to solve the problem is much bigger if compared to the present work. This is due to the fact that in [40] the nodal values for bending moments are obtained numerically by integrating the stress along the plate thickness (using a Gauss scheme). Therefore, one RVE has to be assigned to each Gauss point defined along the plate thickness and related to a particular plate node. In the present work, only one RVE has to be defined for each cell node, as the stresses are constant along the plate thickness. Thus, the BEM formulation adopted in the present work to model the macro-continuum is completely different to the one considered in [40], as now we deal with the plate stretching problem. In the present work, we intend to show that the FEM formulation considered in [40] to model the micro-scale, can be coupled to others BEM macro-continuum formulations, you have only to assign one RVE to each point where the stresses evaluation is required and to change the macro-continuum formulation according to the problem to be analysed. After solving the RVE equilibrium problem, the stress vector and constitutive tensor can be computed for a particular point and the analysis continues according to the macro-continuum formulation. Besides, in [40] we have shown the BEM as a good alternative to perform multi-scale analysis for the plate problem. In the present work, we intend to show that the BEM is also a good alternative to perform the multi-scale analysis of the plate stretching problem, as almost all the works already published about multi-scale modelling consider only the FEM to model all the scales. In a future work, the authors intend to propose a multi-scale model where only the BEM will be considered to model both the macro and micro scales.

2. BEM formulation for modelling the macro-continuum

2.1. The non-linear two-dimensional problem

The non-linear plate stretching analysis, that represents the macro-continuum problem in the present work, is modelled by a BEM non-linear formulation discussed in details in [37]. To define the plate stretching problem, let us consider a flat plate of thickness t, external boundary Γ and domain Ω referred to a Cartesian system of co-ordinates with x_1 and x_2 axes laying on its middle surface and x_3 being the axis perpendicular to that plane. It is assumed that the plate supports only loads acting in the x_1 and x_2 directions over the plate middle surface. The variables related to the plate stretching problem are: the in-plane tractions ($\dot{p}_n$ and $\dot{p}_s$), in-plane displacements ($\dot{u}_n$ and $\dot{u}_s$), being n and s the local co-ordinate system, with n and s referring to the boundary normal and tangential directions, respectively. As the present work deals with non-linear analysis, all variables are expressed in rates, i.e., $(\dot{x}) = dx/dt$, their time derivatives. The basic equilibrium equations for the plate stretching problem will be omitted here, but they can be found in several works [32–37].

The membrane internal forces are obtained by integrating the Cauchy stresses $\dot{\sigma}_{ij}$ across the plate thickness which for the two-dimensional problem, considering plane stress results into:

$$\dot{N}_{ij} = \int_{-t/2}^{t/2} \dot{\sigma}_{ij} dz = t\dot{\sigma}_{ij} \quad i,j = 1,2 \tag{1}$$

Note that in a conventional non-linear analysis, $\dot{\sigma}_{ij}$ is obtained after verifying the constitutive model for a particular point of the plate. In the multi-scale analysis the dissipative processes occur in the RVE that represents the microstructure, as discussed in Section (3). Thus in a multi-scale analysis $\dot{\sigma}_{ij}$ is obtained after solving the RVE equilibrium problem and the membrane internal force $\dot{N}_{ij}$ rates evaluated by replacing $\dot{\sigma}_{ij}$ into Eq. (1).

As this work only deals with small strain problems the total strain will be split into its elastic and inelastic parts, $\dot{\varepsilon}^e_{ij}$ and $\dot{\varepsilon}^p_{ij}$ respectively, as follows:

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij} + \dot{\varepsilon}^p_{ij} \quad i,j = 1,2 \tag{2}$$

By applying the Hooke's law the stress tensor rate $\dot{\sigma}_{ij}$ (as well as the force $\dot{N}_{ij}$) is related to the elastic part $\dot{\varepsilon}^e_{ij}$ of the strain tensor rate and the membrane force predictor $\dot{N}^e_{ij}$ (often defined as elastic trial used in non-linear algorithms) related to the total strain $\dot{\varepsilon}_{ij}$. Thus, the forces $\dot{N}^e_{ij}$ can be written in terms of the total displacements derivatives as follows:

$$\dot{N}^e_{ij} = \mu\left[\dot{u}_{i,j} + \dot{u}_{j,i}\right] + \frac{2\mu\nu'}{1-2\nu'}\dot{u}_{k,k}\delta_{ij} \quad i,j,k = 1,2 \tag{3}$$

where δ_{ij} is the Kronecker delta, μ is the shear elastic modulus, $\nu' = \nu/(1+\nu)$, being ν the Poisson's ratio; if the plane stress is considered (case of the present paper), μ has to be weighted by the plate thickness t.

Therefore, the inelastic membrane force rate $\dot{N}^p_{ij}$ is defined as:

$$\dot{N}^p_{ij} = \dot{N}^e_{ij} - \dot{N}_{ij} \tag{4}$$

Observe that although an elasto-plastic criterion has been adopted in the present work for describing the dissipative process in the RVE, the presented BEM formulation used to model the macro-continuum can be also used with other kind of criteria.

2.2. BEM integral representations and algebraic equations

The non-linear formulation is obtained from Betti's reciprocal theorem (see more details in [37]) which in the case of the plate stretching problem can be written as:

$$\int_\Omega \varepsilon^*_{ijk}\dot{N}_{jk} d\Omega = \int_\Omega N^*_{ijk}\dot{\varepsilon}_{jk} d\Omega - \int_\Omega \varepsilon^*_{ijk}\dot{N}^{(p)}_{jk} d\Omega \quad i,j,k = 1,2 \tag{5}$$

where the values with superscript * are related to the fundamental problem, being i the direction of the fundamental load.

By integrating Eq. (5) by parts twice one obtains the well known representation of in-plane displacements written for internal and