Engineering Analysis with Boundary Elements 56 (2015) 81–89

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Numerical solution of nonlinear fractional integro-differential equations by hybrid functions

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ARTICLE INFO

Article history:
Received 20 July 2014
Received in revised form
5 November 2014
Accepted 3 February 2015
Available online 5 March 2015

Keywords:
Hybrid functions
Nonlinear fractional integro-differential equations
Bernoulli polynomials
Caputo derivative
Numerical solution

ABSTRACT

In this paper, a new numerical method for solving nonlinear fractional integro-differential equations is presented. The method is based upon hybrid functions approximation. The properties of hybrid functions consisting of block-pulse functions and Bernoulli polynomials are presented. The Riemann–Liouville fractional integral operator for hybrid functions is given. This operator is then utilized to reduce the solution of the nonlinear fractional integro-differential equations to a system of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

1. Introduction

Fractional differential equations (FDEs) and fractional integro-differential equations (FIDEs) have drawn increasing attention and interest due to their important applications in science and engineering (see for example [1–3]).

Many mathematical modelings of various physical phenomena contain non-linear fractional-order Volterra integro-differential equations, such as heat conduction in materials with memory [4]. Moreover, these kinds of equations always arise in fluid dynamics, biological models and chemical kinetics [5,6]. Generally speaking, the analytical solutions of most FIDEs are not easy to obtain. Therefore, seeking numerical solutions of these equations becomes more and more important [7]. Recently, several numerical methods to solve FIDEs have been given, such as the variational iteration method [8,9], homotopy perturbation method [10–12], Adomian's decomposition method [13], homotopy analysis method [14] and collocation method [15,16].

The available sets of orthogonal functions can be divided into three classes. The first class includes sets of piecewise constant basis functions (e.g., block-pulse, Haar, and Walsh). The second class consists of sets of orthogonal polynomials (e.g., Chebyshev, Laguerre, and Legendre). The third class is the set of sine–cosine functions in the Fourier series. Orthogonal functions have been used when dealing with various problems of the dynamical systems. The main advantage of using orthogonal functions is that they reduce the dynamical system problems to those of solving a system of algebraic equations by using the operational matrices of differentiation or integration. These matrices can be uniquely determined based on the particular orthogonal functions. Special attention has been given to applications of the Walsh functions, rational Legendre functions, rationalized Haar functions, Legendre wavelets and semi-orthogonal wavelets [17–22]. The Bernoulli polynomials and Taylor series are not based on orthogonal functions. Nevertheless, they possess the operational matrix of integration.

In recent years, the hybrid functions consisting of the combination of block-pulse functions with Legendre polynomials, Chebyshev polynomials, Taylor series, Lagrange polynomials or Bernoulli polynomials [23–32] have been shown to be a mathematical power tool for discretization of selected problems. Among these hybrid functions, the hybrid functions of block-pulse and Bernoulli polynomials have been shown to be computationally more effective [31–33]. To the best of our knowledge, none of these hybrid functions have been applied for problems with fractional order differential equations. Furthermore, for solving fractional order differential equations by cosine and sine (CAS), Chebyshev, Haar, or Legendre wavelets, the operational matrices for fractional order (OMFFO) of these wavelets are calculated in [34–37], respectively. For obtaining the OMFFO, these wavelets were first expanded into block-pulse functions, then OMFFO of block-pulse functions was used for calculating OMFFO for CAS, Chebyshev, Haar and

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http://dx.doi.org/10.1016/j.enganabound.2015.02.002
0955-7997/

Legendre wavelets in [34–37], respectively. It is noted that none of these wavelets calculated OMFFO directly.

In the present paper, a new numerical method for solving the system of fractional integro-differential equations of the following form is presented:

$$F_1(t,f(t),D^{q_0}f(t),D^{q_1}f(t),\ldots,D^{q_r}f(t))=\lambda F_2(t,f(t),\int_0^t \kappa(t,s)G(s,f(s))\,ds), \tag{1}$$

with initial conditions

$$f^{(k)}(0)=d_k,\quad k=0,1,\ldots,m_0-1, \tag{2}$$

where $q_0 \ge q_1 \ge \cdots \ge q_r \ge 0,\ m_k-1<q_k\le m_k,\ 0\le t\le 1$ and $\lambda \in R$.

The method is based upon hybrid functions approximation. These hybrid functions, which consist of the hybrid of block-pulse functions and Bernoulli polynomials, are given. We then obtain the Riemann–Liouville fractional integral operator for the hybrid of block-pulse functions and Bernoulli polynomials. This operator is then utilized to reduce the solution of Eq. (1) with initial conditions in Eq. (2) to the solution of algebraic equations.

The outline of this paper is as follows: In Section 2, we introduce some necessary definitions and mathematical preliminaries of fractional calculus. In Section 3, we describe some properties of the hybrid of block-pulse functions and Bernoulli polynomials required for our subsequent development. In Section 4, we derive the Riemann–Liouville fractional integral operator for the hybrid of block-pulse functions and Bernoulli polynomials. Section 5 is devoted to the numerical method for solving Eq. (1) with initial conditions in Eq. (2). In Section 6, we report our numerical findings and demonstrate the accuracy of the proposed numerical scheme by considering nine numerical examples.

2. Preliminaries and notations

2.1. The fractional derivative and integral

There are various definitions of fractional derivative and integration. The widely used definition of a fractional derivative is the Caputo definition, and a fractional integration is the Riemann–Liouville definition.

Definition 1. Caputo's fractional derivative of order q is defined as [38]

$$(D^q f)(t)=\frac{1}{\Gamma(n-q)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}}\,ds,\quad n-1<q\le n,\ n\in\mathbb{N},$$

where $q>0$ is the order of the derivative and n is the smallest integer greater than q.

Definition 2. The Riemann–Liouville fractional integral operator of order q is defined as [38]

$$I^q f(t)=\begin{cases}\dfrac{1}{\Gamma(q)}\int_0^t \dfrac{f(s)}{(t-s)^{1-q}}\,ds=\dfrac{1}{\Gamma(q)}t^{q-1}*f(t), & q>0,\\ f(t), & q=0,\end{cases} \tag{3}$$

where $t^{q-1}*f(t)$ is the convolution product of t^{q-1} and $f(t)$.

The Caputo derivative and Riemann–Liouville integral satisfy the following properties [39]:

$$I^\alpha(D^\alpha h(t))=h(t)-\sum_{k=0}^{n-1}h^k(0)\frac{t^k}{k!}, \tag{4}$$

if $\alpha\in R,\ n-1<\alpha\le n,\ n\in N$, then

$$D^\alpha(h(t))=I^{n-\alpha}D^n h(t). \tag{5}$$

3. Hybrid of block-pulse functions and Bernoulli polynomials

Hybrid functions $b_{nm}(t),\ n=1,2,\ldots,N,\ m=0,1,\ldots,M$, are defined on the interval $[0,t_f)$ as [33]

$$b_{nm}(t)=\begin{cases}\beta_m\left(\dfrac{N}{t_f}t-n+1\right), & t\in\left(\dfrac{n-1}{N}t_f,\dfrac{n}{N}t_f\right),\\ 0 & \text{otherwise},\end{cases} \tag{6}$$

where n and m are the order of block-pulse functions and Bernoulli polynomials, respectively. In Eq. (6), $\beta_m(t),\ m=0,1,2,\ldots$, are the Bernoulli polynomials of order m, which can be defined by [40]

$$\beta_m(t)=\sum_{k=0}^{m}\binom{m}{k}\alpha_{m-k}t^k, \tag{7}$$

where $\alpha_k,\ k=0,1,\ldots,m$, are Bernoulli numbers [33]. These polynomials satisfy the following formula [41]:

$$\beta_m(1-x)=(-1)^m\beta_m(x). \tag{8}$$

3.1. Function approximation

If $f\in L^2[0,1]$ and the best approximation of f by using the hybrid of block-pulse functions and Bernoulli polynomials is $P_M^N f$, then [33]

$$f\simeq P_M^N f=\sum_{m=0}^{M}\sum_{n=1}^{N}c_{nm}b_{nm}(t)=C^T B(t), \tag{9}$$

where

$$C^T=[c_{10},c_{20},\ldots,c_{N0},c_{11},c_{21},\ldots,c_{N1},\ldots,c_{1M},c_{2M},\ldots,c_{NM}], \tag{10}$$

and

$$B^T(t)=[b_{10}(t),b_{20}(t),\ldots,b_{N0}(t),b_{11}(t),b_{21}(t),\ldots,b_{N1}(t),\ldots,b_{1M}(t),b_{2M}(t),\ldots,b_{NM}(t)]. \tag{11}$$

4. Riemann–Liouville fractional integral operator for hybrid of block-pulse functions and Bernoulli polynomials

We now derive the operator I^α for $B(t)$ in Eq. (11) given by

$$I^\alpha B(t)=\overline{B}(t,\alpha), \tag{12}$$

where

$$\overline{B}(t,\alpha)=[I^\alpha b_{10}(t),\ldots,I^\alpha b_{N0}(t),I^\alpha b_{11}(t),\ldots,I^\alpha b_{N1}(t),\ldots,I^\alpha b_{1M}(t),I^\alpha b_{2M}(t),\ldots,I^\alpha b_{NM}(t)]^T. \tag{13}$$

To obtain $I^\alpha b_{nm}(t)$, we use the Laplace transform. By using Eq. (6), we have

$$b_{nm}(t)=\mu_{(n-1)/N}(t)\beta_m(Nt-n+1)-\mu_{n/N}(t)\beta_m(Nt-n+1), \tag{14}$$

where $\mu_c(t)$ is unit step function defined as

$$\mu_c(t)=\begin{cases}1, & t\ge c,\\ 0, & t<c.\end{cases}$$

By taking the Laplace transform from Eq. (14) and using Eq. (8), we get

$$\begin{aligned}L[b_{nm}(t)]&=e^{-((n-1)/N)s}L\left[\beta_m\left(N\left(t+\frac{n-1}{N}\right)-n+1\right)\right]\\&\quad-e^{-(n/N)s}L\left[\beta_m\left(N\left(t+\frac{n}{N}\right)-n+1\right)\right],\\&=e^{-((n-1)/N)s}L[\beta_m(Nt)]-(-1)^m e^{-(n/N)s}L[\beta_m(-Nt)].\end{aligned}$$