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Local integration of 2-D fractional telegraph equation via moving least squares approximation

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ABSTRACT

In this paper, a classical type of two-dimensional time-fractional telegraph equation defined by Caputo sense for $(1<\alpha<2)$ is analyzed by an approach based on the Galerkin weak form and moving least squares (MLS) approximation subject to given appropriate initial and Dirichlet boundary conditions. In the proposed method, which is a kind of the Meshless local Petrov–Galerkin (MLPG) method, meshless Galerkin weak form is applied to the interior nodes while the meshless collocation method is used for the nodes on the boundary, so the Dirichlet boundary condition is imposed directly. In MLPG method, it does not require any background integration cells so that all integrations are carried out locally over small quadrature domains of regular shapes, such as circles or squares. The moving least squares approximation is proposed to construct shape functions. Two numerical examples are presented and satisfactory agreements are achieved.

1. Introduction

The fractional derivative and fractional differential equations have been implemented to describe some phenomena in physics and engineering, such as boundary layer effects in ducts, allometric scaling laws in biology and ecology, colored noise, dielectric polarization, electromagnetic waves, electrode–electrolyte polarization, fractional kinetics, quantitative finance, quantum evolution of complex systems, power-law phenomenon in fluid and complex network, and viscoelastic mechanics [33,36]. Furthermore, the fractional telegraph equation (typical fractional diffusion-wave equation) has been applied into signal analysis for transmission, the modeling of the reaction diffusion, propagation of electrical signals and the random walk of suspension flows, etc. [12,32].

The classical telegraph equation is the outcome of the variational connection between the voltage wave and the current wave on the well-proportioned transmission line (therefore it is also called the transmission line equation). However, the classical telegraph equation could not well represent the abnormal diffusion phenomena during the finite long transmits progress, where the voltage wave or the current wave possibly exists [2,42–44]. So, it is necessary to study the fractional telegraph equation, including the time and (or) space fractional derivatives. The present paper considers the following time-fractional telegraph equation of order $(1<\alpha<2)$:

$$\frac{\partial^\alpha u(\mathbf{x},t)}{\partial t^\alpha}+\gamma_1\frac{\partial^{\alpha-1}u(\mathbf{x},t)}{\partial t^{\alpha-1}}+\gamma_2 u(\mathbf{x},t)=\gamma_3\Delta u+f(\mathbf{x},t),\quad \mathbf{x}\in\Omega,\ t\in[0,T], \tag{1}$$

subject to compatible initial conditions

$$u(\mathbf{x},0)=\varphi(\mathbf{x}),\quad \frac{\partial u}{\partial t}(\mathbf{x},0)=\psi(\mathbf{x}),\quad \mathbf{x}\in\Omega, \tag{2}$$

and the boundary conditions

$$u(\mathbf{x},t)=g_0(y,t)\quad \text{for } x=-1,\quad u(\mathbf{x},t)=g_1(y,t)\quad \text{for } x=1,\ \mathbf{x}\in\Omega,\ t\in[0,T], \tag{3}$$

$$u(\mathbf{x},t)=h_0(x,t)\quad \text{for } y=-1,\quad u(\mathbf{x},t)=h_1(x,t)\quad \text{for } y=1,\ \mathbf{x}\in\Omega,\ t\in[0,T], \tag{4}$$

where $\mathbf{x}=(x,y)$ is the spatial variable, $\Omega=[-1,1]^2=\{(x,y):-1\le x, y\le 1\}$ and, γ_1, γ_2 and γ_3 are constants. Also, $f(\mathbf{x},t)$ is the source function with sufficient smoothness and $\varphi(\mathbf{x})$, $\psi(\mathbf{x})$, $g_0(y,t)$, $g_1(y,t)$, $h_0(x,t)$ and $h_1(x,t)$ are given continuous functions. Moreover, in Eq. (1), the time-fractional derivatives are in the sense of Caputo which is defined by

$$D_t^\alpha F(t)=\begin{cases}\dfrac{1}{\Gamma(k-\alpha)}\int_0^t(t-\xi)^{k-\alpha-1}F^{(k)}(\xi)\,\mathrm{d}\xi, & k-1<\alpha<k,\ t>0,\\ F^{(k)}(t), & \alpha=k.\end{cases} \tag{5}$$

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There are three classes of meshless methods: meshless methods based on weak forms such as the element free Galerkin (EFG) method [7,8,38], meshless methods based on collocation techniques (strong forms) such as the meshless collocation method based on radial basis functions (RBFs) [27,21,16,1] and meshless methods based on the combination of weak forms and collocation technique. Due to the ill-conditioning of the resultant linear systems in RBF-collocation method, various approaches are proposed to evade this problem, Refs. [22–25] being among them.

In the literature, several meshless weak form methods have been reported such as diffuse element method (DEM) [35], smooth particle hydrodynamic (SPH) [9,10], the reproducing kernel particle method (RKPM) [30], boundary node method (BNM) [34], partition of unity finite element method (PUFEM) [31], finite sphere method (FSM) [11], boundary point interpolation method (BPIM) [18] and boundary radial point interpolation method (BRPIM) [19,39,41]. Liu applied the concept of MLPG and developed meshless local radial point interpolation (MLRPI) method [28,26,13]. The weak forms are used to derive a set of algebraic equations through a numerical integration process using a set of quadrature domain that may be constructed globally or locally in the domain of the problem. In the global weak form methods, global background cells are needed for numerical integration in computing the algebraic equations. To avoid the use of global background cells, a so-called local weak form is adopted to develop the meshless local Petrov–Galerkin (MLPG) method [3–6,14,15,17,40,37]. When a local weak form is used for a field node, the numerical integrations are carried out over a local quadrature domain defined for the node, which also be the local domain where the test (weight) function is defined. The local domain usually has a regular and simple shape for an internal node (such as sphere and square), and the integration is done numerically within the local domain. Hence the domain and boundary integrals in the weak form methods can easily be evaluated over the regularly shaped sub-domains and their boundaries.

In this paper, we focus on the numerical solution of Eqs. (1)–(4) using a kind of MLPG method which is based on the Galerkin weak form and moving least squares (MLS) approximation. Two illustrative examples are given so that the convergence occurs with respect to both time discretization and total number of nodes covering spatial domain.

2. The MLS approximation scheme

A meshless method uses a local interpolation or approximation to represent the trial function with the values of the unknown variable at some nodal points. In the current work, the moving least squares (MLS) approximation is used. Consider a sub-domain Ω_s, the neighborhood of a point $\mathbf{x}$ and denoted as the domain of definition (or support) of the MLS approximation for the trial function at $\mathbf{x}$, which is located in the problem domain Ω (see Fig. 1). To approximate the distribution of function u in Ω_s, over a number of randomly located nodes $\mathbf{x}_i, i=1,2,\ldots n$, the moving least squares approximant $u^h(\mathbf{x})$ of u, $\forall \mathbf{x}\in\Omega_s$, can be defined by

$$u^h(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{a}(\mathbf{x}) \quad \forall \mathbf{x}\in\Omega_s, \tag{6}$$

where $\mathbf{p}^T(\mathbf{x}) = [p_1(\mathbf{x}),\ p_2(\mathbf{x}),\ldots,p_m(\mathbf{x})]$ is a complete monomial basis of order m, and $a(\mathbf{x})$ is a vector containing coefficients $a_j(\mathbf{x})$, $j=1,2,\ldots m$, which are functions of the space coordinates $\mathbf{x}$. $p_j(\mathbf{x})$ is monomial in the space coordinate $x^T=[x,y]$, and m is the number of polynomial basis functions. The $p_j(\mathbf{x})$ is built using Pascal's triangle and a complete basis is usually preferred. The linear basis functions are given by

$$\mathbf{P}^T(\mathbf{x}) = \{1,x,y\}, \quad m=3, \tag{7}$$

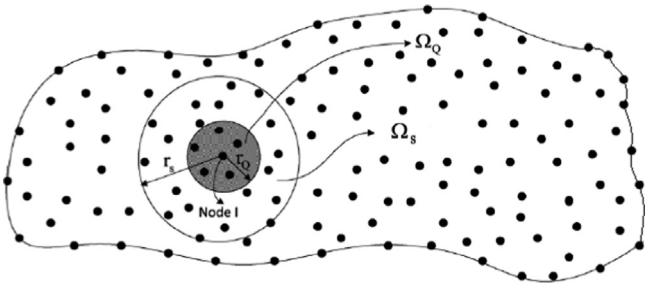

Fig. 1. Node I is an interior node. Ω_s and Ω_Q are local support and local quadrature domains, respectively.

and the quadratic basis functions are

$$\mathbf{P}^T(\mathbf{x}) = \{1,x,y,x^2,y^2,xy\}, \quad m=6. \tag{8}$$

The coefficient vector $\mathbf{a}(\mathbf{x})$ is determined by minimizing a weighted discrete L_2 norm, defined as

$$J(x) = \sum_{i=1}^{n} w_i(\mathbf{x})[p^T(\mathbf{x}_i)\mathbf{a}(\mathbf{x}) - \hat{u}_i]^2$$
$$= [\mathbf{P}.\mathbf{a}(\mathbf{x}) - \hat{\mathbf{u}}]^T.W.[\mathbf{P}.\mathbf{a}(\mathbf{x}) - \hat{\mathbf{u}}], \tag{9}$$

where $w_i(\mathbf{x})$ is the weight function associated with the node i, with $w_i(\mathbf{x})>0$ for all $\mathbf{x}$ in the support of $w_i(\mathbf{x})$, $\mathbf{x}_i$ denotes the value of $\mathbf{x}$ at node i, n is the number of nodes in Ω_s for which the weight functions $w_i(\mathbf{x})>0$, the matrices $\mathbf{P}$ and $\mathbf{W}$ are defined as

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}^T(\mathbf{x}_1) \\ \mathbf{p}^T(\mathbf{x}_2) \\ \vdots \\ \mathbf{p}^T(\mathbf{x}_n) \end{pmatrix}_{n\times m}, \quad \mathbf{W} = \begin{pmatrix} w_1(\mathbf{x}) & \ldots & 0 \\ \vdots & \ldots & \vdots \\ 0 & \ldots & w_n(\mathbf{x}) \end{pmatrix}$$

and $\hat{\mathbf{u}}^T = [\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n]$. Here it should be noted that $\hat{u}_i,\ i=1,2,\ldots,n$, in Eq. (9) are the fictitious nodal values, and not the nodal values of the unknown trial function $u^h(\mathbf{x})$ in general. The stationarity of J in Eq. (9) with respect to $\mathbf{a}(\mathbf{x})$ leads to the following linear relation between $\mathbf{a}(\mathbf{x})$ and $\hat{\mathbf{u}}$:

$$\mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\hat{\mathbf{u}}, \tag{10}$$

where the matrices $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are defined by

$$\mathbf{A}(\mathbf{x}) = \mathbf{P}^T\mathbf{W}\mathbf{P} = \mathbf{B}(\mathbf{x})\mathbf{P} = \sum_{i=1}^{n} w_i(\mathbf{x})\mathbf{p}(\mathbf{x}_i)\mathbf{p}^T(\mathbf{x}_i), \tag{11}$$

$$\mathbf{B}(\mathbf{x}) = \mathbf{P}^T\mathbf{W} = [w_1(\mathbf{x})\mathbf{p}(\mathbf{x}_1), w_2(\mathbf{x})\mathbf{p}(\mathbf{x}_2), \ldots, w_n(\mathbf{x})\mathbf{p}(\mathbf{x}_n)]. \tag{12}$$

The MLS approximation is well defined only when the matrix $\mathbf{A}$ in Eq. (10) is non-singular. It can be seen that this is the case if and only if the rank of $\mathbf{P}$ equals m. A necessary condition for a well-defined MLS approximation is that at least m weight functions are non-zero (i.e. $n>m$) for each sample point $\mathbf{x}\in\Omega$ and that the nodes in Ω_s will not be arranged in a special pattern such as on a straight line. Here a sample point may be a nodal point under consideration or a quadrature point.

Solving for $\mathbf{a}(\mathbf{x})$ from Eq. (10) and substituting it into Eq. (6) give a relation which may be written as the form of an interpolation function similar to that used in FEM, as

$$u^h(\mathbf{x}) = \Phi^T(\mathbf{x}).\hat{\mathbf{u}} = \sum_{i=1}^{n} \phi_i(\mathbf{x})\hat{u}_i, \quad \mathbf{x}\in\Omega_s, \tag{13}$$

where $u^h(\mathbf{x}_i)\equiv u_i$ is not essentially equal to $\hat{u}_i$ and

$$\Phi^T(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x}) \tag{14}$$