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Estimation of effective elastic moduli of random structure composites by the method of fundamental solutions

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ABSTRACT

One considers linearly elastic composite media, which consist of a homogeneous matrix containing a statistically homogeneous random set of aligned homogeneous heterogeneities of non-canonical shape. Effective elastic moduli as well as the first statistical moments of stresses in the phases are estimated through the averaged boundary integrals over the inclusion boundaries. The modified popular micromechanical models are based on the numerical solution for one inhomogeneity inside the infinite matrix loaded by remote homogeneous effective field. This solution is obtained by a meshfree method based on fundamental solutions basis functions for a transmission problem in linear elasticity. The problem here addressed, consists in computing the displacement and traction fields of an elastic object, which has piecewise constant Lamé coefficients, from a given displacement (or stress) field on the infinity. The main properties of the method are analyzed and illustrated with several numerical simulations in 2D domains.

1. Introduction

The prediction of the behavior of composite materials in terms of the mechanical properties of constituents and their microstructure is a central problem of micromechanics, which is evidently reduced to the estimation of stress fields in the constituents. Appropriate, but by no means exhaustive, references for the estimation of effective elastic moduli of statistically homogeneous media are provided by the reviews [1–8]. It appears today that variants of the effective medium method [9,10] and the mean field method [11,12] are the most popular and widely used methods. Recently a new method has become known, namely the multiparticle effective field method (MEFM) was put forward and developed (see for references Buryachenko [6]). The MEFM is based on the theory of functions of random variables and Green's functions. Within this method one constructs a hierarchy of statistical moment equations for conditional averages of the stresses in the inclusions. The hierarchy is then cut by introducing the notion of an effective field. This way the interaction of different inclusions is taken into account. The popular schemes of micromechanical analysis are based on numerical solutions for estimation of stress (or strain) distribution tensor for single (at least) inclusion inside infinite matrix subjected to the so-called effective field.

Obtaining analytical solutions is not feasible in general even for a finite number of interacting particles, so various numerical methods have been developed, mostly based on the finite element analysis (FEA) and boundary integral equation BIE) technique. A distinct advantage of the BIE compared to the FEA is that the BIEs require meshing only the boundary surface of computational domain as opposed to the entire 3-D domain for FEA. In the BIE singular forces distributed, e.g., on the surface of particles, depend on the external field, thus yielding an integral equation for the singularity strengths. The most popular formulation is the direct one where the variables of interest are the surface displacement and traction fields that lead to a set of ill-conditioned linear systems on discretization. The indirect formulation (e.g. the completed double layer boundary element method, see [13,14]) results in a set of integral equations of the second kind, and therefore is numerically well-behaved. Although this formulation is more complicated than the direct method, however, it leads to an unconditionally convergent iterative scheme.

Despite the FEM's and BEM's popularity, there are many problems (e.g. meshing surface in 3D, computing of singular integrals) where it is desirable to improve the efficiency of traditional methods. In the alternative (proposed by Trefftz), the trial functions must satisfy the governing equation exactly and the error in the satisfaction of the boundary conditions is minimized. As a consequence, it is interesting to develop some alternative so-called meshless methods (see [15] for an overview and historical background on the subject) of the local boundary integral equation, boundary knot method, boundary collocation method, non-dimensional dynamic influence functions method, and the

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method of fundamental solutions (MFS) belonging to a boundary method for solving boundary value problems, which can be recognized as a discrete type of the indirect boundary element method with a concentrated source instead of distribution (see for references [16–19]). In this study, we will bring the MFS systematically for a single inclusion inside the infinite homogeneous matrix subjected to homogeneous loading. The MFS belonging to the BIE technique has a very attractive option since it is truly meshless, simple to program, and is able to take into account sharp changes in geometry. The MFS and related methods over the last few years have found extensive application in computing solutions to a broad range of problems [20,21].

In the MFS the solution is approximated by a finite linear combination of fundamental solutions (FS, assumed to be explicitly defined) with respect to source points which are positioned outside the solution domain. The unknown coefficients in the mentioned linear approximation can be determined by matching the boundary conditions in a variety of ways, the simplest (and most popular) being by collocation on a set of physical boundary points. In contrast to domain discretization methods such as the finite element (FEM), the MFS is a boundary method which means that only the boundary of the solution domain needs to be considered. Moreover, the MFS is a boundary-type meshless method which does not need a boundary element mesh, either for purposes of interpolation of the trial and test functions of the solution variables, or for the integration, and only a set of source points is required for the discretization of the problem being analyzed. However, unlike the BEM, no potentially troublesome integration is required in the MFS due to the placement of source points outside the solution domain when the singular integrals are avoided. At last MFS is adaptive in the sense that it can take into account sharp changes in the solution and in the geometry of the domain and can easily handle complex boundary conditions. Despite the mentioned advantages, a few disadvantages are that the positioning of the source points is preassigned and also the resulting system of algebraic equations is ill-conditioned that leads to oscillation of the convergence curve of the numerical solution when a large number of source points are used. Optimization of source points allocation substantially reduces computational time while some regularization methods, such as the damped singular value decomposition, the truncated singular value decomposition or the Tikhonov [22] regularization, can be used to mitigate the ill-conditioned effect (see [23]). The next step is checking the approximation in other (not collocation) points at the physical boundary (or interface) surface. The strain and stress field variables outside the boundary can be obtained directly when the derivatives are analytically calculated from the MFS expansion representation over the source points with the coefficients found by matching the boundary condition at the given collocation points at the physical surface.

Thus, the stress distributions inside a single inclusions inside infinite matrix are assumed to be found. This solution then can be incorporated into the one or another general framework of analytical micromechanics for self-consistent estimations of the so-called effective field (see for details Buryachenko [6]). However, all mentioned methods are based on the effective field hypothesis (EFH, even if the term "*effective field hypothesis*" was not indicated) according to which each inclusion is located inside a homogeneous so-called effective field (see for references [6]). Effective field hypothesis is apparently the most fundamental, most prospective, and most exploited concept of micromechanics. This concept has directed a development of micromechanics over the last sixty years and made a contribution to their progress incompatible with any another concept. The idea of effective field was added by the hypothesis of "*ellipsoidal symmetry*" for the distribution of inclusions attributed to Willis [24]. However, Buryachenko [25,26] has proved that the EFH is a central one and other concepts play a satellite role providing the conditions for application of the EFH.

The paper is organized as follows. In Section 2 we present the basic field equations of linear elasticity, notations, statistical description of the composite microstructure as well as representation of the effective properties through the surface integrals over the inclusion boundaries. In Section 3, one presents the method of fundamental solution (MFS) adapted to the solution for one homogeneous noncanonical inclusion inside the infinite homogeneous matrix. The known micromechanical method of the effective field (MEF, see for references [6]) and Mori–Tanaka method (MTM, [11,12]) is presented in Section 4 with the stress concentration factors and effective compliances expressed though the boundary integral of statistical averages over the boundary inclusions. In Section 5 we estimate the numerical errors of both the different versions of the MFS and the different choices of the source sets. The numerical evaluation of the effective Young modulus is shown for statistically homogeneous composites reinforced by aligned identical homogeneous heterogeneities of noncanonical shape.

2. Preliminaries

Let a full space R^d with a space dimensionality d ($d=2$ and $d=3$ for 2-D and 3-D problems, respectively) contains a homogeneous matrix $v^{(0)}$ and, in general, a statistically inhomogeneous set $X=(v_i)$ of heterogeneities v_i with indicator functions V_i and bounded by the closed smooth surfaces $\Gamma_i{:=}\partial v_i$ $(i=1,2,\ldots)$ defined by the relations $\Gamma_i(\mathbf{x})=0$ $(\mathbf{x}\in\Gamma_i)$, $\Gamma_i(\mathbf{x})>0$ $(\mathbf{x}\in v_i)$, and $\Gamma_i(\mathbf{x})<0$ $(\mathbf{x}\notin v_i)$. It is assumed that the heterogeneities can be grouped into components (phases) $v^{(q)}$ $(q=1,2,\ldots,N)$ with identical mechanical and geometrical properties (such as shape, size, orientation, and microstructure of heterogeneities).

2.1. Basic equations

Let a linear elastic body occupy an open bounded domain $w\subset R^d$ with a smooth boundary Γ and with an indicator function W and space dimensionality d ($d=2$ and $d=3$ for 2-D and 3-D problems, respectively). The domain w contains a homogeneous matrix $v^{(0)}$ and a statistically inhomogeneous set $X=(v_i,V_i,\mathbf{x}_i)$ of inclusions v_i with indicator functions V_i and centers $\mathbf{x}_i$. It is assumed that the inclusions can be grouped into component (phase) $v^{(1)}$ with identical mechanical and geometrical properties (such as shape, size, orientation, and microstructure of inclusions). For the sake of definiteness, in the 2-D case we will consider a plane-strain problem. At first no restrictions are imposed on the elastic symmetry of the phases or on the geometry of the inclusions.[1]

The problem is governed by the local equations of elastostatics of composites

$$\nabla\boldsymbol{\sigma}(\mathbf{x})=\mathbf{0}, \qquad (1)$$

$$\boldsymbol{\sigma}(\mathbf{x})=\mathbf{L}(\mathbf{x})\boldsymbol{\varepsilon}(\mathbf{x}) \quad \text{or } \boldsymbol{\varepsilon}(\mathbf{x})=\mathbf{M}(\mathbf{x})\boldsymbol{\sigma}(\mathbf{x}), \qquad (2)$$

$$\boldsymbol{\varepsilon}(\mathbf{x})=[\nabla\otimes\mathbf{u}+(\nabla\otimes\mathbf{u})^{\top}]/2, \quad \nabla\times\boldsymbol{\varepsilon}(\mathbf{x})\times\nabla=\mathbf{0}, \qquad (3)$$

where $(\cdot)^{\top}$ denotes transposition, $\otimes$ and $\times$ are the tensor and vector products. $\mathbf{L}(x)$ and $\mathbf{M}(x)\equiv L(x)^{-1}$ are the known stiffness and compliance fourth-order tensors, and the common notation for contracted products has been employed.

[1] It is known that for 2-D problems the plane-strain state is only possible for material symmetry no lower than orthotropic (see e.g. [27]) that will be assumed hereafter in the 2-D case.