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Boundary element formulation of the Mild-Slope equation for harmonic water waves propagating over unidirectional variable bathymetries

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ABSTRACT

This paper presents a boundary element formulation for the solution of the Mild-Slope equation in wave propagation problems with variable water depth in one direction. Based on Green's function approximation proposed by Belibassakis [1], a complete fundamental-solution kernel is developed and combined with a boundary element scheme for the solution of water wave propagation problems in closed and open domains where the bathymetry changes arbitrarily and smoothly in a preferential direction. The ability of the proposed formulation to accurately represent wave phenomena like refraction, reflection, diffraction and shoaling, is demonstrated with the solution of some example problems, in which arbitrary geometries and variable seabed profiles with slopes up to 1:3 are considered. The obtained results are also compared with theoretical solutions, showing an excellent agreement that demonstrates its potential.

1. Introduction

Wave propagation in variable-depth waters is a problem of significant importance in coastal engineering with applications in the design and maintenance of harbors, coastal defense works and hydrodynamic and sediment transportation studies. It is well known that the transmission of linear waves in intermediate and deep waters can be reproduced by the elliptical Mild-Slope Equation MSE) which was derived by Berkhoff [2] in the early 1970s. The MSE considers simultaneously the effects of diffraction, refraction, reflection and shoaling of linear water surface waves and it is formally valid for slowly varying sea bed slopes, i.e. $\nabla h \ll kh$, being h the water depth and k the wave number. The validity of the MSE has been evaluated by Tsay and Liu [3] demonstrating that it produces accurate results for bottom slopes up to 1:1 when waves are propagating perpendicularly to the bathymetry contour lines. Nevertheless, Booij [4] verified that, for general directions of wave propagation, the MSE is able to provide acceptable accuracy for bottom profiles with slopes up to 1:3, enough for practical applications.

Some extensions of the MSE have been proposed in subsequent works. For example, a time-dependent extension of the MSE was derived by Kirby [5] for the case of waves propagating over ripple beds. Also, an Extended Mild-Slope Equation (EMSE) was proposed by Massel [6] that includes higher-order terms, providing a better accuracy for more complicated bathymetries. Energy dissipation effects, such as wave breaking and bottom friction, were included in [7]. Chamberlain and Porter [8] suggested a Modified Mild-Slope Equation (MMSE), later improved by Porter and Staziker [9], which retains the second order terms discarded by Berkhoff in the formulation of the MSE. On the other hand, Suh et al. [10] derived a time-dependent equation for wave propagation on rapidly varying topography and Chandrasekera et al. [11] included terms for relatively steep and rapidly undulating bathymetries. Later, Lee et al. [12] presented an hyperbolic MSE for rapidly varying topography, followed by the works of Copeland [13] and Massel [6] in the same direction. Finally, the recent works of Hsu et al. [14] and Li et al. [15,16] considered higher-order bottom effect terms to account for a rapidly varying topography and wave energy dissipation in the surf zone. Basically, all these formulations introduce higher-order terms in the MSE due to the bottom effects, usually proportional to the square of the bottom slope or the bottom curvature.

In general, the MSE represents the basic framework for the simulation of surface wave transmission problems in variable water depths and different numerical solution procedures have been proposed in the literature since the pioneering work of Berkhoff [2].

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Traditionally, the MSE has been solved using the Finite Element Method (FEM) [17] and the Finite Difference Method (FDM), where we can include the works of Li and Anastasiou [18], Panchang and Pearce [19]. Nevertheless, finite difference schemes and the finite element method present a common deficiency; open and partially reflecting boundary conditions are difficult to represent. These deficiencies have been studied by many authors, like Chen et al. [20,21] using hybrid FEM formulations, together with the initial proposals of Berkhoff [17] and Tsay et al. [3,22] including bottom friction effects. For the closing boundary conditions, Bettess and Zienkiewicz [23] and Lau and Ji [24] used infinite elements in the outer regions. Dirichlet to Neumann (DtN) boundary conditions were proposed by Givoli et al. [25–27] as an analytical procedure to reproduce exact non-reflecting boundary conditions in some particular cases. This idea, was followed by Bonet [28] to derive the discrete non-local (DNL) boundary condition. More rudimentary iterative methods have also been proposed to define absorbing boundary conditions; see Beltrami et al. [29], Steward and Panchang [30], Chen [31] or Liu et al. [32], among others. It is important to mention that a boundary element formulation of the MSE for open domains and variable bathymetry, would be able to palliate the drawbacks of FEM, providing a better approximation for the simulation of absorbing boundaries.

The MSE problem has also been solved using the Boundary Element Method (BEM). Boundary element techniques prove to be very accurate in wave refraction–diffraction problems with open domains, presenting the additional benefit that the radiation condition to infinity is automatically satisfied. In order to improve the solution of the FEM schemes, Hauguel [33] and Shaw and Falby [34] first coupled FEM and BEM. Hamanaka [35] proposed a genuine BEM based boundary condition for open, partial reflection and incident-absorbing boundaries. At the same time, Isaacson and Qu [36] introduced a boundary integral formulation to reproduce the wave field in harbors with partial reflecting boundaries and Lee et al. [37,38] included the effect of incoming random waves. The Dual Reciprocity Boundary Element Method (DRBEM) has been used to model wave run-ups by Zhu [39]. Later, this technique was extended to model internal regions with variable depth surrounded by exterior regions with constant bathymetry [40–43]. More recently, Naserizabeh et al. [44] proposed a coupled BEM-FDM formulation to solve the MSE in unbounded problems.

In this context, this paper presents a BEM formulation for the MSE in wave propagation problems with variable water depth in one direction. Based on Green's function approximation proposed by Belibassakis [1], a complete fundamental-solution kernel is developed and combined with a boundary element scheme for the solution of water wave propagation problems in closed and open domains where the bathymetry changes arbitrarily and smoothly in a preferential direction. This particular case is of high practical interest, because the bathymetric lines can usually be considered straight and parallel to the coast-line. A BEM formulation of the MSE for variable bathymetry not only extends the range of applications of the BEM for the solution of coastal engineering problems but also, combined with the FEM and used as a matching condition, offers the possibility of modeling very accurately the radiation condition to deeper waters.

The paper is organized as follows. Section 2 first reviews the formulation of the MSE. In Section 3, the fundamental solution of the MSE for variable water depth is approximated in the frequency domain. The mathematical and numerical principles of the BEM for wave scattering problems are covered in Sections 4. Section 5 is dedicated to the validation of the proposed BEM formulation through the solution of wave propagation problems in variable water depth. Finally, Section 6 closes with the conclusions.

2. The Mild-Slope equation

The classical MSE [2,17] is obtained from the linear wave theory using a Cartesian coordinate system with the (x,y)-plane located on the quiescent water surface and the z direction pointing upwards. Under the assumption of potential flow and integrating the velocity potential in the vertical direction with appropriated boundary conditions, the velocity potential of the water surface can be represented in the form:

$$\Phi(x,y,t) = \phi(x,y)e^{-i\omega t}, \tag{1}$$

being i the imaginary unit and t the time variable. This potential has to satisfy the homogeneous MSE, that may be written as:

$$\nabla \cdot (cc_g \nabla \phi) + \omega \frac{c_g}{c} \phi = 0, \tag{2}$$

where $\nabla = (\partial_x, \partial_y)$ is the gradient operator, c is the wave velocity and c_g the group velocity. The water depth function $h(x,y)$, wave number k and angular frequency ω of the waves are related by the dispersion equation:

$$\omega^2 = gk \tanh(kh), \tag{3}$$

being g the gravitational acceleration ($g = 9.81\ \mathrm{m/s^2}$). This means that, for a fixed frequency and variable bathymetry, the wave number $k(x,y)$ is a function of the local water depth.

The MSE can be simplified introducing the following change of variable due to Bergmann [45]:

$$\phi = \frac{1}{\sqrt{cc_g}} \hat{\phi}, \tag{4}$$

a relation that transforms (2) into a Helmholtz equation:

$$\nabla^2 \hat{\phi} + \hat{k}^2 \hat{\phi} = 0, \tag{5}$$

with a modified wave number $\hat{k}(x,y)$ given by

$$\hat{k}^2(x,y) = k^2 - \frac{\nabla^2 \sqrt{cc_g}}{\sqrt{cc_g}}, \tag{6}$$

that is a known function of the wave characteristics and the local water depth.

Note that this approach is also valid for treating the same problem in the framework of the MMSE. Simply by modifying the expression of the wave number (6), including additional effects associated with higher-order contributions of bottom slope and curvature, we obtain the MMSE model that extends the applicability of the MSE.

3. Fundamental solution for variable wave number

Based on Green's function of Belibassakis [1], in this section we develop a fundamental solution of the Helmholtz problem (5) for the particular case of an unidirectional variable bathymetry like the one described in Fig. 1. Taking the x-axis in the same direction than the variation of the water-depth $h = h(x)$, a modified wave-

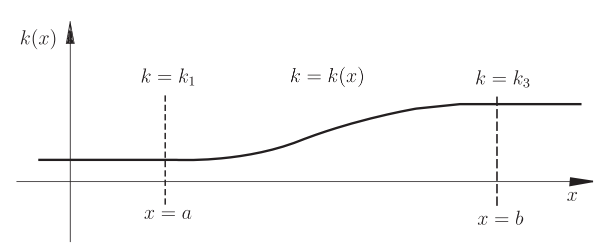

Fig. 1. Wave number variation in the x-direction for a fixed wave frequency due to a monotonically decreasing water depth profile $h(x)$. Wave number is higher where water depth is lower as dictated by the dispersion relation.