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A multiple-scale Pascal polynomial triangle solving elliptic equations and inverse Cauchy problems

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ABSTRACT

The polynomial expansion method is a useful tool to solve partial differential equations (PDEs). However, the researchers seldom use it as a major medium to solve PDEs due to its highly ill-conditioned behavior. We propose a single-scale and a multiple-scale Pascal triangle formulations to solve the linear elliptic PDEs in a simply connected domain equipped with complex boundary shape. For the former method a constant parameter R_0 is required, while in the latter one all introduced scales are automatically determined by the collocation points. Then we use the multiple-scale method to solve the inverse Cauchy problems, which is very accurate and very stable against large noise to 20%. Numerical results confirm the validity of the present multiple-scale Pascal polynomial expansion method.

1. Introduction

There are many papers which are concerned with the numerical solutions of elliptic type boundary value problems (BVPs) for the computational applications in several areas, such as, Liu [1–6], and Pradhan et al. [7]. Nowadays the meshless and mesh reduction methods are the main stream of numerical computation methods, to name a few, Zhu et al. [8,9], Atluri and Zhu [10,11], Atluri et al. [12], Atluri and Shen [13], Cho et al. [14], Jin [15], and Li et al. [16]. These methods always lead to nonlinear algebraic equations (NAEs), when one applied them to solve nonlinear partial differential equations (PDEs). Many collocation techniques which are special cases of the meshless local Petrov–Galerkin (MLPG) method [10,11], which together with the expansions by different basis-functions were employed to solve the elliptic type BVPs; see, for example, Cheng et al. [17], Hu et al. [18], Algahtani [19], Tian et al. [20], Hu and Chen [21], and Libre et al. [22]. Li et al. [23] gave a very detailed description of the collocation Trefftz method. Basically, the above bases expansion methods are effective for linear problems.

The purpose of this paper is to develop a quite powerful polynomial expansion algorithm, having the advantage of easy numerical implementation, and having a great flexibility applied to most linear elliptic type BVPs defined in arbitrary plane domain.

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We begin with the following linear elliptic equation:

$$\Delta u(x,y) = F(x,y,u,u_x,u_y), \quad (x,y) \in \Omega, \tag{1}$$

$$u(x,y) = H(x,y), \quad (x,y) \in \Gamma, \tag{2}$$

where Δ is the Laplacian operator, Γ is the boundary of the problem domain Ω, and F and H are given functions with F being linear on (u, u_x, u_y).

The polynomial interpolation is an ill-posed problem and it makes the interpolation by higher-order polynomials not being easy to be numerically implemented. In order to overcome these difficulties, Liu and Atluri [24] have introduced a characteristic length into the high-order polynomials expansion, which improved the numerical accuracy for the applications to solve some ill-posed linear problems. At the same time, Liu et al. [25] have developed a multi-scale Trefftz-collocation Laplacian conditioner to deal with the ill-conditioned linear systems. This concept of multi-scale Trefftz-collocation method has been later employed by Chen et al. [26] to solve the sloshing wave problem. Liu [27] has proposed a multi-scale half-order polynomial interpolation method, and Kuo et al. [28] have used the modified two characteristic lengths Pascal triangle method to solve inverse heat source problem. In this paper we extend the work by Kuo et al. [28], and propose a new multiple-scale expansion technique by high-order polynomials, which can overcome the above-mentioned ill-conditioned behavior.

This paper is arranged as follows. In Section 2 we introduce a simple modification of the Pascal triangle expansion method by considering a characteristic length. Then according to the concept of equilibrated matrix we introduce a multiple-scale into the Pascal triangle expansion method, of which the introduced scales are fully determined by the collocation points. The numerical examples for the direct problems are solved in Section 4. The inverse Cauchy problems are described in Section 5, while the numerical examples are given in Section 6. Finally, we draw some conclusions in Section 7.

2. A modified polynomial expansion method

The use of polynomial expansion as a trial solution of PDE is simple and is straightforward to derive the required algebraic equations after a suitable collocation in the problem domain. However, it is seldom used as a major numerical tool to solve linear PDEs. The main reason is that the resultant linear algebraic equations (LAEs) are often highly ill-conditioned. How to reduce the condition number of the linear system becomes an important issue in the application of polynomials expansion to PDEs.

The elements in following polynomial matrix:

$$\begin{bmatrix} 1 & y & y^2 & \cdots & y^{m-1} & y^m \\ x & xy & xy^2 & \cdots & xy^{m-1} & xy^m \\ x^2 & x^2y & x^2y^2 & \cdots & x^2y^{m-1} & x^2y^m \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ x^m & x^my & x^my^2 & \cdots & x^my^{m-1} & x^my^m \end{bmatrix} \tag{3}$$

are often used to expand the solution of $u(x,y)$. If the elements are restricted in the left-upper triangle then such an expansion is known as the Pascal triangle expansion:

$$\begin{array}{c} 1 \\ x \;\; y \\ x^2 \;\; xy \;\; y^2 \\ x^3 \;\; x^2y \;\; xy^2 \;\; y^3 \\ x^4 \;\; x^3y \;\; x^2y^2 \;\; xy^3 \;\; y^4 \\ x^5 \;\; x^4y \;\; x^3y^2 \;\; x^2y^3 \;\; xy^4 \;\; y^5 \\ \cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots \end{array} \tag{4}$$

Therefore, the solution $u(x,y)$ is expanded by

$$u(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij} x^{i-j} y^{j-1}, \tag{5}$$

where the coefficients c_{ij} are to be determined, whose number of all elements is $n = m(m+1)/2$. The highest order of the above polynomial is $m-1$.

Because x and y in the problem domain Ω may be an arbitrarily large quantity, the above expansion would lead to a divergence of the powers x^m and y^m. Thus according to the suggestion by Liu and Atluri [24] we can employ the following modified polynomial expansion method, involving a normalized length scale R_0, to express the solution:

$$u(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij} \left(\frac{x}{R_0}\right)^{i-j} \left(\frac{y}{R_0}\right)^{j-1}, \tag{6}$$

where the coefficients c_{ij} are to be determined, whose number of all elements is $n = m(m+1)/2$. The highest order of the above polynomial is $m-1$. Here we use a modified Pascal triangle to expand the solution, where $R_0 > 0$ is the characteristic length of the plane domain we consider. Basically we need $\Omega \in [-R_0, R_0] \times [-R_0, R_0]$.

From Eq. (6) it is straightforward to write

$$u_x(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij}(i-j)R_0 \left(\frac{x}{R_0}\right)^{i-j-1} \left(\frac{y}{R_0}\right)^{j-1}, \tag{7}$$

$$u_y(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij}(j-1)R_0 \left(\frac{x}{R_0}\right)^{i-j} \left(\frac{y}{R_0}\right)^{j-2}, \tag{8}$$

$$\Delta u(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij}R_0^2 \left[(i-j)(i-j-1)\left(\frac{x}{R_0}\right)^{i-j-2}\left(\frac{y}{R_0}\right)^{j-1} + (j-1)(j-2)\left(\frac{x}{R_0}\right)^{i-j}\left(\frac{y}{R_0}\right)^{j-3}\right]. \tag{9}$$

Inserting these equations into Eqs. (1) and (2), and selecting n_1 and n_2 collocation points on the boundary and in the domain, to satisfy the boundary condition and the field equation, respectively, we can obtain a system of LAEs to solve the n coefficients c_{ij}. The above method is a single-scale Pascal triangle expansion method.

3. A multiple-scale Pascal triangle

In order to obtain an accurate solution of the linear elliptic PDE and inverse Cauchy problem by using the modified Pascal triangle polynomial expansion method, we have to develop more effective and accurate solution method to solve these LAEs by reducing the condition numbers. Instead of the single-scale expansion we consider a novel multiple-scale Pascal triangle expansion of $u(x,y)$ by

$$u(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij} s_{ij} x^{i-j} y^{j-1}, \tag{10}$$

where the scales s_{ij} will be determined below.

From Eq. (5) it is straightforward to write

$$u_x(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij}(i-j) x^{i-j-1} y^{j-1}, \tag{11}$$

$$u_y(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij}(j-1) x^{i-j} y^{j-2}, \tag{12}$$

$$\Delta u(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij}\left[(i-j)(i-j-1)x^{i-j-2}y^{j-1} + (j-1)(j-2)x^{i-j}y^{j-3}\right]. \tag{13}$$

Inserting these equations into Eqs. (1) and (2), and selecting n_1 and n_2 collocation points on the boundary and in the domain, to satisfy the boundary condition and the field equation, respectively, we can obtain a system of LAEs to solve the n coefficients c_{ij}. It is convenient to express the resulting LAEs in terms of a matrix-vector product form:

$$\mathbf{A}\mathbf{c} = \mathbf{b}. \tag{14}$$

Usually, Eq. (14) is an over-determined system for that we may collocate more points to generate more equations, which are used to find n coefficients in $\mathbf{c}$ with $n \ll n_c$.

First the coefficients c_{ij} used in the expansion (5) can be expressed as an n-dimensional vector $\mathbf{c}$ with components $c_k, k = 1, \ldots, n$.

Then for a generic point $(x,y) \in \Omega$ the term $u(x,y)$ can be expressed as an inner product of a vector $\mathbf{a}$ with $\mathbf{c}$, i.e.,

$$u(x,y) = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 & \ldots \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{a}^{\mathrm{T}}\mathbf{c}. \tag{15}$$

Similarly, for a generic point $(x,y) \in \Omega$ the term $\Delta u(x,y)$ can be expressed as an inner product of a vector $\mathbf{d}$ with $\mathbf{c}$, where the