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A BIEM using the Trefftz test functions for solving the inverse Cauchy and source recovery problems

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ABSTRACT

In this paper we develop a global domain/boundary integral equation method for the Laplace and Poisson equations, which is based on the Green's second identity. A derived global relation links the source term to the Dirichlet and Neumann boundary conditions into a single integral equation in terms of the Trefftz test functions. By suitably choosing the Trefftz test functions, which are not the usual Green functions as that used in the conventional boundary integral method, the present boundary integral equation method (BIEM) can find the unknown boundary conditions for the inverse Cauchy problems very well. Even under a large noise to 10% and the data over-specified in a 25% portion of the whole boundary, the recovered result is still accurate. The inverse source problems of the Poisson equation are resolved numerically by using the BIEM which is stable and effective for strongly ill-posed case with a large noise being imposed on the supplementary data.

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1. Introduction

The inverse Cauchy problem is to solve the boundary value problem of elliptic type partial differential equations given by some over-specified Cauchy data on a partial portion of the boundary, which is proved to have a unique solution if the solution exists. However, the problem of numerical instability causes the inverse of the original operator not available. For example, the exact solution

$$u(x,y) = n^{-2}\ \sin(nx)\sinh(ny)$$

does not become small for any nonzero y, even the initial condition $n^{-1}\ \sin(nx)$ can be arbitrarily small by increasing n. In the Hardmard sense, the solution does not depend continuously on the initial data. To treat this kind ill-posed problem, many techniques were proposed; among them the most famous one is the Tikhonov's regularization technique, which transforms the original problem into a constrained minimization problem.

The use of electrostatic image in the non-destructive testing of metallic plates leads to an inverse Cauchy problem for the Laplace equation in two-dimension. In order to detect the unknown shape of the inclusion within a conducting metal, the over-determined Cauchy data, for example the voltage and current, are imposed on the accessible exterior boundary [1–3]. This amounts to solving an inverse Cauchy problem from available data on partial boundary. The Cauchy problem is difficult to be solved both numerically and analytically, since its solution does not depend continuously on the given data as just mentioned.

In the past decades there are many numerical methods proposed to solve the Cauchy problems [4–10], to name a few. Among the many numerical methods, the schemes based on iteration have been developed by Jourhmane and Nachaoui [11,12], Essaouini et al. [13], Nachaoui [14], and Jourhmane et al. [15]. Liu [16] has applied a modified collocation Trefftz method in the inverse Cauchy problem in a circular domain. In [17,18], a similar method has been named the Fourier regularization method. Liu [19] has developed a modified Trefftz method by a simple collocation technique to treat the inverse Cauchy problem of Laplace equation in arbitrary plane domain. Liu and Kuo [20], Liu et al. [21] and Liu and Chang [22] have proposed using the spring-damping regularization techniques to treat the inverse Cauchy problems. Then Liu and Atluri [23] and Liu [24] used a better post-conditioning collocation Trefftz method to solve the inverse Cauchy problems.

In this work we also ponder the inverse source problem of the Poisson equation, which arises in many branches of sciences and engineering, e.g. crack identification, electromagnetic theory, and geophysical prospecting. There are some papers for identifying the unknown source in the Poisson equation by utilizing the regularization methods. For example, Ohe and Ohnaka [25] identified the

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unknown point source with the logarithmic potential; Nara and Ando [26] identified the unknown point source using the projective method; Hon et al. [27] identified the unknown point source with Green's function; Farcas et al. [28] identified the unknown source using the dual reciprocity boundary element method; Jin and Marin [29] identified the unknown source of one variable using the method of fundamental solutions (MFS) Yang and Fu [30,31] used the truncation method and modified regularization method for identifying an unknown source of one variable in the Poisson equation.

The remaining portion of this paper is arranged as follows. In Section 2 we introduce a global domain/boundary integral method based on the Green's theorem and the adjoint operator, which results in a reciprocity gap functional to extract unknown boundary data from over-specified measurements. In Section 3 we choose a suitable set of the Trefftz test functions to derive a linear system for the inverse Cauchy problem in a rectangle, whose numerical examples are given in Section 4. In Section 5 we derive a linear system for the inverse Cauchy problem in a simply-connected domain and give numerical tests. The inverse source problem of the Poisson equation is addressed in Section 6, where some numerical examples are given, and finally the conclusions are drawn in Section 7.

2. Boundary integral equation method

2.1. The inverse Cauchy problem in a rectangle

We consider an inverse Cauchy problem given as follows:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \quad 0<x<a, \quad 0<y<b, \tag{1}$$

$$u(x,b)=h_1(x), \quad 0\le x\le a, \tag{2}$$

$$u_y(x,b)=h_2(x), \quad 0\le x\le a, \tag{3}$$

$$u(0,y)=u_0(y), \quad u(a,y)=u_a(y), \quad 0\le y\le b, \tag{4}$$

where $h_1(x)$, $u_0(y)$ and $u_a(y)$ are given functions, $h_2(x)$ is an over-specified function, and the subscript y denotes the partial differential with respect to y.

In this inverse Cauchy problem we suppose that the Neumann datum $h_2(x)$ in Eq. (3) is over-specified, such that we can determine the unknown functions $f(x)$ and $g(x)$ on the bottom:

$$u(x,0)=f(x), \quad u_y(x,0)=g(x), \quad 0\le x\le a. \tag{5}$$

2.2. Green's second identity

Before embarking the derivation of Green's second identity for Laplace equation, we introduce the Laplacian operator:

$$\Delta u(x,y)=u_{xx}+u_{yy}. \tag{6}$$

Lemma 1 (*Green's Theorem in the plane*). *Let Ω be a bounded region in the plane (x,y) with a counter-clockwise contour Γ consists of finitely many smooth curves. Let $F_1(x,y)$ and $F_2(x,y)$ be functions that are differentiable in Ω and continuous on $\overline{\Omega}$. Then*

$$\int\int_\Omega \left[\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right] dx\, dy=\oint_\Gamma (F_1\, dx+F_2\, dy). \tag{7}$$

Inserting

$$F_1=vu_y-uv_y, F_2=uv_x-vu_x, \tag{8}$$

and by using Lemma 1 we can prove Green's second identity for the Laplacian operator.

Theorem 1 (*Green's second identity*). *Let Ω be a bounded region in the plane (x,y) with a counter-clockwise contour Γ consists of finitely many smooth curves. Let $u(x,y)$ and $v(x,y)$ be functions that are twice differentiable in Ω and continuous on $\overline{\Omega}$. Then*

$$\int\int_\Omega (u\Delta v-v\Delta u)\, d\sigma=\oint_\Gamma (uv_n-vu_n)\, ds, \tag{9}$$

where $d\sigma=dx\, dy$ is an area element in the plane and the subscript n denotes the normal derivative with respect to $\mathbf{n}=(dy/ds,-dx/ds)$.

Proof. The proof of Green's second identity is available in text books, and we omit it.□

Theorem 2 (*Global relation*). *For the inverse Cauchy problem in Eqs. (1)–(5), f(x) and g(x) satisfy the following global relation:*

$$\begin{aligned}\oint_\Gamma (uv_n-vu_n)\, ds=&\int_0^a [g(x)v(x,0)-f(x)v_y(x,0)]\, dx\\ &+\int_0^b [u_a(y)v_x(a,y)-v(a,y)u_x(a,y)]\, dy\\ &-\int_0^a [h_2(x)v(x,b)-h_1(x)v_y(x,b)]\, dx\\ &-\int_0^b [u_0(y)v_x(0,y)-v(0,y)u_x(0,y)]\, dy=0\end{aligned} \tag{10}$$

for any function v with $\Delta v=0$.

Proof. Inserting $\Delta u=0$ and $\Delta v=0$ into Eq. (9), integrating along the contour $\Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4=\{0\le x\le a, y=0\}\cup\{x=a, 0\le y\le b\}\cup\{0\le x\le a, y=b\}\cup\{x=0, 0\le y\le b\}$, and inserting the corresponding conditions in Eqs. (2)–(4) we can prove this theorem.□

3. The numerical algorithm of boundary integral equation method

In Theorem 2 we can choose a simple function $v(x,y)$, such that Eq. (10) can be easily used to solve $f(x)$ and $g(x)$. For this purpose we can take

$$v(x,y)=\sin\frac{k\pi x}{a}\exp\left(\frac{-k\pi y}{a}\right), \tag{11}$$

which is a solution of the Laplace equation with $k\in\mathbb{N}$ being a positive integer. Inserting

$$\begin{aligned}&v(0,y)=0, \quad v(a,y)=0, \quad v(x,0)=\sin\frac{k\pi x}{a},\\ &v_y(x,0)=-\frac{k\pi}{a}\sin\frac{k\pi x}{a}\end{aligned} \tag{12}$$

into Eq. (10) we can derive

$$\begin{aligned}\int_0^a \left[\frac{k\pi}{a}f(x)+g(x)\right]\sin\frac{k\pi x}{a}\, dx=&\int_0^b [u_0(y)v_x(0,y)-u_a(y)v_x(a,y)]\, dy\\ &+\int_0^a [h_2(x)v(x,b)\\ &-h_1(x)v_y(x,b)]\, dx=:e_k,\end{aligned} \tag{13}$$

where e_k is a different constant for different k, and

$$\begin{aligned}&v(x,b)=\sin\frac{k\pi x}{a}\exp\left(\frac{-k\pi b}{a}\right),\\ &v_x(0,y)=\frac{k\pi}{a}\exp\left(\frac{-k\pi y}{a}\right),\\ &v_x(a,y)=\frac{k\pi}{a}\cos(k\pi)\exp\left(\frac{-k\pi y}{a}\right),\end{aligned}$$

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