Engineering Analysis with Boundary Elements 62 (2016) 186–192

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

The natural boundary integral equation of the orthotropic potential problem

Huan-Lin Zhou*, Yu Tian, Bo Yu, Zhong-Rong Niu

School of Civil Engineering, Hefei University of Technology, Hefei 230009, PR China

A R T I C L E I N F O

Article history:
Received 4 July 2015
Received in revised form
4 September 2015
Accepted 7 September 2015
Available online 1 October 2015

Keywords:
Natural boundary integral equation
Orthotropic potential problems
BEM
Nearly singular integrals
Analytical integral regularization algorithm

A B S T R A C T

The governing equation of two-dimensional orthotropic potential problem is transformed into standard Laplace equation by the coordinate transformation method. Then a novel potential derivative boundary integral equation termed natural boundary integral equation (NBIE) is established for the two-dimensional orthotropic potential problem. The NBIE reduces the singularity by one order when compared with the conventional potential derivative boundary integral equation (CDBIE). Thus the potential derivative of the two-dimensional orthotropic potential problem can be computed more accurately by using the NBIE at the same boundary mesh. Furthermore, after using the analytical integral regularization algorithm of nearly singular integrals, the NBIE can obtain more accurate potential derivatives of interior points which are very close to the boundary than the CDBIE. Numerical examples verify the accuracy and efficiency of the present method.

1. Introduction

The orthotropic potential problem is an important subject in the practical engineering. For example, the heat property coefficients of materials change with the coordinate direction in heat conduction problems. Many researchers have studied orthotropic potential problems systematically by the boundary element method (BEM). Brebbia and Chang [1] analyzed the seepage problems by the BEM for the inhomogeneous soil medium. Perez and Wrobel [2] proposed an integral equation formulation for the analysis of orthotropic potential problems. Tanaka et al. [3] used the BEM to analyze the transient heat conduction problems in orthotropic bodies. Zhou et al. [4] developed a completely new analytical integral algorithm to treat nearly singular integrals for two-dimensional orthotropic potential problems with thin bodies.

It is clear that the boundary potential and its normal derivative can be obtained by applying the conventional BIE. If the potential derivative is required, the traditional method is directly to differentiate the potential function along the boundary. However, this process will degrade the accuracy by one order. In other words, it reduces the advantage of the BEM. Many researchers sought to establish the derivative BIE, and numerous papers about the derivative BIE had been published. Chen et al. [5] presented a derivative BIE based on the dual integral formulation to analyze the seepage flow under a dam with sheet piles. Ghosh et al. [6] derived a new kind of derivative BIE to study the two-dimensional linear elasticity problem, in which the tangential derivative of the displacement is taken as an unknown variable instead of the displacement itself. Choi and Kwak [7] presented another kind of derivative BIE to solve the two-dimensional potential problem. Recently, Huang and Liu [8] applied the fast multi-pole boundary element method to evaluate the deflection of the plate and its derivative inside the plate.

Furthermore, no matter what form the derivative BIE is, the singularity is always raised by one order. It means that the strongly singular integral in the conventional BIE will be transformed into the hypersingular integral in the derivative BIE. The literature concerning singular integrals was too much extensive and reviewed in [9,10]. Sladek et al. [11] brought together a comprehensive treatment of the principal strategies developed to deal with the various kinds of singular integrals. They [12] also gave some introductory notes on singular integrals in the first chapter of their book. Gray et al. [13] transformed the hypersingular integral into the line integral along the contour of the element, and then the numerical quadrature was employed to compute the line integral. The subtraction method [14,15] was used by subtracting and adding back a term for the hypersingular integrand at a load point so that the hypersingular integrals were transformed into the strongly singular integrals by the Stokes's theorem. Liu [16] demonstrated the line integral method which transformed nearly singular or hypersingular integrals to sums of nearly weakly singular integrals and nonsingular line integrals. Aimi et al. [17] used the Galerkin method to compute the singular or hypersingular

* Corresponding author.
E-mail address: zhouhl@hfut.edu.cn (H.-L. Zhou).

http://dx.doi.org/10.1016/j.enganabound.2015.09.002
0955-7997/

integral equations associated with two-dimensional boundary value problems defined on domains whose boundaries have piecewise smooth parametric representations. Recently, Li and Huang [18] proposed an improved reformulation of the Burton–Miller method and regularized the hypersingular integrals by using a new singularity subtraction technique. Kress [19] presented a fully discrete collocation method for dealing with the hypersingular integral equation arising from the double-layer approach for the solution of Neumann boundary value problems. Chen and Hong [20,21] proposed the dual integral equations for problems with a degenerate boundary and developed the dual BEM. The regularization of hypersingular integrals occurring in dual BEM was summarized in detail by them [22]. Recently, Chen et al. [23] derived dual boundary integral equations for the N-dimensional Laplace problems with a smooth boundary by using the contour approach surrounding the singularity. Yu [24,25] proposed the natural BEM which contains the hypersingular integral operator in the Dirichlet to Neumann (DtN) map. Yu [26] discussed the natural integral operator and natural boundary element method for harmonic problem over exterior elliptic domain. The dual BEM and natural BEM all naturally resulted in hypersingular integrals. It seems that the hypersingular integrals can be treated by using the above methods. But for some special cases, especially the hypersingular integrals in a corner or discontinuous points of the physical, the convinced results are still not obtained. In fact, the derivative BIE contains both the strongly singular and the hypersingular integrals, the sum of the two integrals is generally existent. However, it is difficult to find the way to determine it.

Niu et al. [27] proposed a displacement derivative BIE termed natural boundary integral equation (NBIE) in the two-dimensional elasticity problems in an unique way. In this derivative BIE, only the strongly singular integral exists and it can be calculated perfectly. Afterwards, Niu and Zhou [28] established another NBIE in isotropic potential problems by using the analogous idea. The new NBIE also has the perfectly performance for solving the two-dimensional isotropic potential problems. In literatures [10–12], Sladek et al. also derived a new important integral representation of potential gradients that the same regularization was done for general anisotropic medium. Moreover, it was done therein more transparently without using any coordinate transformation.

In this paper, the authors attempt to extend the NBIE of isotropic potential problems to analyze orthotropic potential problems. Firstly, the governing equation of two-dimensional orthotropic potential problem is transformed into standard Laplace equation by the coordinate transformation method. Then, the NBIE is introduced to solve the potential derivative on boundary and in domain.

2. The governing equation of two-dimensional orthotropic potential problems

The two-dimensional orthotropic potential problem can be described by the following governing equation (Fig. 1).

$$k_1\frac{\partial^2 u}{\partial y_1^2}+k_2\frac{\partial^2 u}{\partial y_2^2}=0 \tag{1}$$

where we assume that the property coefficients k_i in the orthotropic direction y_i are constant values. The fundamental solution of Eq. (1) can be written as follows:

$$u^*=\frac{1}{2\pi\sqrt{k_1k_2}}\ln\frac{1}{r} \tag{2}$$

in which $r=\sqrt{(y_1^s-y_1)^2/k_1+(y_2^s-y_2)^2/k_2}$, y_i^s and y_i are the coordinates of the source point y^s and the field point y, respectively.

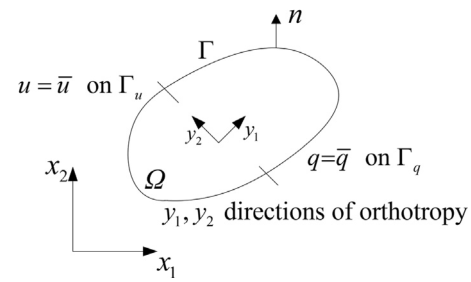

Fig. 1. Orthotropic potential problems.

Let n_1 and n_2 denote the components of the outward normal vector n of the boundary Γ. For the linear element, $n_1=(y_{F2}-y_{I2})/l$, $n_2=-(y_{F1}-y_{I1})/l$, y_{Ii} and y_{Fi} are the coordinates of the element starting and ending points, respectively. l is the length of the element. The boundary normal potential derivative can be given by

$$q^*=k_1\frac{\partial u^*}{\partial y_1}n_1+k_2\frac{\partial u^*}{\partial y_2}n_2 \tag{3}$$

The boundary conditions of the two-dimensional orthotropic potential problem can be expressed as

$$\begin{cases} u(y)=\bar{u}(y) & \textbf{on } \Gamma_u \\ q(y)=k_1\frac{\partial u}{\partial y_1}n_1+k_2\frac{\partial u}{\partial y_2}n_2=\bar{q}(y) & \textbf{on } \Gamma_q \end{cases} \tag{4}$$

3. Coordinate transformation method

In order to employ the NBIE of the isotropic potential problems to solve the orthotropic potential problems, we assume that $\hat{y}_i=y_i/\sqrt{k_i}$ (the superscript " ^ " means that this variable belongs to the new coordinate system, similarly hereinafter). The following relations can be obtained

$$\frac{\partial u}{\partial y_i}=\frac{\partial u}{\partial \hat{y}_i}\frac{\partial \hat{y}_i}{\partial y_i}=\frac{1}{\sqrt{k_i}}\frac{\partial u}{\partial \hat{y}_i} \tag{5}$$

$$\frac{\partial^2 u}{\partial y_i^2}=\frac{1}{k_i}\frac{\partial^2 u}{\partial \hat{y}_i^2} \tag{6}$$

Substituting the two above relations into Eq. (1) and considering $\hat{u}(\hat{y})=u(y)$, the governing Eq. (1) is turned into Laplace equation

$$\frac{\partial^2 \hat{u}}{\partial \hat{y}_1^2}+\frac{\partial^2 \hat{u}}{\partial \hat{y}_2^2}=0 \tag{7}$$

The boundary conditions in the new coordinate system are given by

$$\begin{cases} \hat{u}(\hat{y})=\bar{u}(\hat{y}) & \textbf{on } \hat{\Gamma}_u \\ \hat{q}(\hat{y})=\frac{\partial \hat{u}}{\partial \hat{y}_1}\hat{n}_1+\frac{\partial \hat{u}}{\partial \hat{y}_2}\hat{n}_2=\bar{q}(\hat{y}) & \textbf{on } \hat{\Gamma}_q \end{cases} \tag{8}$$

noticing that $\hat{\Gamma}_u+\hat{\Gamma}_q=\hat{\Gamma}$.

Comparing the $\hat{q}(\hat{y})$ in Eq. (8) with the $q(y)$ in Eq. (4), we can find the following conversion

$$\begin{aligned} \hat{q}\hat{l} &= \frac{\partial \hat{u}}{\partial \hat{y}_1}(\hat{y}_{F2}-\hat{y}_{I2})+\frac{\partial \hat{u}}{\partial \hat{y}_2}(\hat{y}_{I1}-\hat{y}_{F1}) \\ &= \sqrt{\frac{k_1}{k_2}}\frac{\partial u}{\partial y_1}(y_{F2}-y_{I2})+\sqrt{\frac{k_2}{k_1}}\frac{\partial u}{\partial y_2}(y_{I1}-y_{F1}) \\ &= k_1\frac{1}{\sqrt{k_1k_2}}\frac{\partial u}{\partial y_1}(y_{F2}-y_{I2})+k_2\frac{1}{\sqrt{k_1k_2}}\frac{\partial u}{\partial y_2}(y_{I1}-y_{F1}) \\ &= ql\frac{1}{\sqrt{k_1k_2}} \end{aligned}$$