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A local meshless collocation method for solving certain inverse problems

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ABSTRACT

In this paper, we propose a meshless scheme based on compactly supported radial basis functions (CS-RBFs) for solving the Cauchy problem of Poisson's equation and the inverse heat conduction problems in 2D. By assuming the unknown boundary condition to be a polynomial function, the inverse problems can be solved using a procedure similar to the process for solving forward problems. We employ Tikhonov regularization technique under L-curve regularization parameter to obtain a stable numerical solution. Numerical results verify the effectiveness and stability of this method.

1. Introduction

Inverse problems arise in scientific, engineering and even medical fields such as non-destructive testing in stress and strain analysis, cardiography, and the heat conduction problem. As we know, these kinds of problems are ill-posed, which means the solutions do not depend continuously on the boundary conditions. Since any small errors caused by the measurement of input data on the boundary or interior of the domain can result in highly amplified errors in the numerical solutions, traditional methods for well-posed forward problems are not suitable for solving inverse problems. Therefore, developments of effective and stable numerical algorithms are essential.

During the last few decades, many numerical methods have been presented for solving inverse problems [1–11]. Among these papers, most of the numerical algorithms are based on the method of fundamental solution (MFS) [1–5,7], the finite difference method (FDM) [6,8], and the finite element method (FEM) [9–11]. However, every method has its own limitations. For FDM and FEM, the cost of generating meshes for three dimensional problems is quite high. Furthermore, the adaptability of FDM to complex domains is poor. Although FEM has better versatility and adaptability to irregular domains, the computational cost in time and space is extremely high for solving large-scale inverse problems. MFS was first applied to solve elliptic boundary value problems by Fairweather and Karageorghis [12]. The pure MFS is limited for solving homogeneous equations when the fundamental solutions are available. Although MFS can be used to solve inhomogeneous problems by combining with the dual reciprocity method (DRM) [13,14], the severe ill-conditioning of the coefficient matrix and the uncertainty for setting the fictitious boundary hinder its application in practical problems.

The meshless methods based on the radial basis functions (RBFs) are of competitive edge, due to their simplicity in selecting interpolation points and high adaptability to domain shape and equation type. Hon and Wu [15] gave the first approach in applying RBFs to solve the Cauchy problem for Laplace equations. Since then, some papers in this area have been published [16–19]. In these papers, the main idea for solving inverse problems is to approximate the solution by a linear combination of RBFs and directly substitute the approximated solution into the governing equation, the boundary conditions, and the over-specified conditions. As we know, the main difficulty in designing an algorithm stems from the ill-posedness of the inverse problem and the ill-conditioning of the coefficient matrix. Furthermore, the condition number of the coefficient matrix increases dramatically with an increase in the number of interpolation points. Therefore, for large-scale problems, where a large number of interpolation nodes are necessary, the coefficient matrix based on the commonly used globally defined RBFs can be dense and highly ill-conditioned. For this reason, compactly supported RBFs (CS-RBFs) which are positive-definite and can result in a sparse matrix are suitable for solving large-scale problems. CS-RBFs have been extensively used for solving forward

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problems. CS-RBFs have been used in the dual reciprocity boundary element method (DRBEM) for solving Poisson's equation [20–22], Stokes Flow problems [23], in MPS-MFS for solving 3D Hemholtz-type equations [24], and in the collocation method for solving shallow water equations [25]. However, to the best of the authors' knowledge, CS-RBFs have not been used for solving inverse problems yet.

In this paper we propose a stable local meshless numerical method based on CS-RBFs for solving 2D inverse problems with a small number of sensors installed inside the domain. We determine the Dirichlet boundary data on an unreachable boundary. Unlike other direct methods [15–19], we design a novel scheme by first assuming the unknown boundary condition to be a polynomial function and then creating equations based on CS-RBFs in an ingenious process. It is worth mentioning that the size and the number of non-zero elements of the coefficient matrix in the proposed method are much smaller than the traditional direct method using CS-RBFs. Thus, the condition number of the coefficient matrix is significantly smaller.

The paper is organized as follows. In Section 2, we briefly review CS-RBFs. In Section 3, we propose the scheme on how to solve inverse problem for Poisson's equation using CS-RBFs. In Section 4, we propose a 2D IHCP algorithm by following the method presented in Section 3 to further verify the stability of the approach. Furthermore, the Tikhonov regularization method with L-curve scheme is applied to obtain a stable solution. In Section 5, the efficiency and stability of the proposed method are tested in comparison with the conventional direct method used in most papers based on the same CS-RBF.

2. CS-RBFs

Radial basis functions are simple and effective tools in approximating multivariate functions. Let $\mathbf{E}=\{\mathbf{e}_j\}_{j=1}^{l}$ be a set of pairwise distinct points in a domain $\Omega\subseteq\mathbb{R}^2$ with associate values $\{f(\mathbf{e}_j)\}_{j=1}^{l}$. For the commonly used global RBFs φ such as Guassians and multiquadrics, the interpolation matrix $\mathbf{A_E}=\left(\varphi(\|\mathbf{e}_k-\mathbf{e}_j\|)\right)_{1\le j,k\le l}$ is non-sparse. To obtain a more accurate solution for inverse problems, we want to use as many points as possible when conditions allow. However, for a large number of interpolation points, the condition number of the coefficient matrix based on the global basis function can be quite large, leading to a loss of stability and numerical accuracy. Furthermore, the cost of matrix inverting and storing $\mathbf{A_E}$ could be enormous. To overcome all these difficulties, compactly supported RBFs (CS-RBFs) have been introduced as local basis functions. The construction of the CS-RBFs was first established by Wu [26], followed by Wendland [27], and later by Buhmann [28]. In this paper we will focus on the CS-RBFs constructed by Wendland [27]. These functions are piecewise polynomial with minimal degree in terms of the given order of smoothness. The interpolation matrix $\mathbf{A_E}$ based on CS-RBF is sparse and positive definite. A list of 2D CS-RBFs is given in Table 1. In this table, the cut-off function $(r)_+$ is defined to be r if $r\ge 0$ and to be zero elsewhere.

In Table 1, the radius of the support of the function has been normalized to 1. In the real application, we can re-scale the function in this table with the support of radius α using $\varphi(r/\alpha)$ for $\alpha>0$. The sparseness of the interpolation matrix $\mathbf{A_E}$ can be suitably adjusted by choosing the scaling factor α. If α is too small, the reproduction quality is poor, while if α is too large, the matrix $\mathbf{A_E}$ is no longer sparse and it will lost its attractiveness in real applications. Hence, a reasonable choice of the scaling factor α is crucial to compromise between the stability and quality of the approximation.

Table 1
Wendland's CS-RBFs in 2D.

$(1-r)_+^2\in C^0$
$(1-r)_+^4(4r+1)\in C^2$
$(1-r)_+^6(35r^2+18r+3)\in C^4$

3. The local meshless method for a stationary inverse heat conduction equation

First, consider the following inverse problem for Poisson's equation:

$$\Delta u(\mathbf{x})=f(\mathbf{x}),\quad \mathbf{x}\in\Omega, \tag{1}$$

$$u(\mathbf{x})=g(\mathbf{x}),\quad \mathbf{x}\in\partial\Omega_1, \tag{2}$$

$$u(\mathbf{x})=h(\mathbf{x}),\quad \mathbf{x}\in\partial\Omega_2, \tag{3}$$

where $\Omega\subseteq\mathbb{R}^2$ is a bounded domain with boundary $\partial\Omega$, and $\partial\Omega=\partial\Omega_1\cup\partial\Omega_2$, $\partial\Omega_1\cap\partial\Omega_2=\varnothing$ and $\partial\Omega_2\ne\varnothing$. Δ is the Laplacian, f and g are given functions, and h is unknown. Note that $\mathbf{x}=(x,y)$. In addition, the over-specified condition is given as follows:

$$u(\mathbf{x}_i^*)=q(\mathbf{x}_i^*),\quad \mathbf{x}_i^*\in\Omega,\quad i=1,2,\ldots,nq, \tag{4}$$

where $\mathbf{x}_i^*$, $i=1,2,\ldots,nq$, are certain interior points at which sensors are fixed in the reachable part of the domain. Therefore, the measured values $q(\mathbf{x}_i^*)$ $(i=1,2,\ldots,nq)$ are known.

In the inverse problem described above, u at any points on boundary $\partial\Omega_2$ and in Ω should be determined. In Section 3, the main idea and process of applying a local meshless method based on CS-RBF to solve this kind of problem will be explained.

Here, we use a polynomial of degree d to approximate the Dirichlet boundary condition on $\partial\Omega_2$, which means

$$h(\mathbf{x})\simeq\hat{h}(\mathbf{x})=b_0+b_1x+b_2y+b_3x^2+b_4xy+b_5y^2+\cdots+b_{d(d+3)/2}y^d. \tag{5}$$

The above polynomial can be represented in vector form:

$$\hat{h}(\mathbf{x})=\mathbf{p}(\mathbf{x})\mathbf{b},\quad \mathbf{x}\in\partial\Omega_2 \tag{6}$$

where

$$\mathbf{p}(\mathbf{x})=(1\ \ x\ \ y\ \ x^2\ \ xy\ \ y^2\ \ \cdots\ \ y^d)$$
$$\mathbf{b}=\left(b_0\ \ b_1\ \ \cdots\ \ b_{d(d+3)/2}\right)^T \tag{7}$$

In order to solve problems (1)–(4), we tentatively assume that $h(\mathbf{x})$ is known as shown in (6), which means $\mathbf{b}$ in (6) is given. Then problems (1)–(4) can be taken as the well-posed forward problem. We apply Kansa's method [29] based on compactly supported RBFs to solve this hypothetical forward problem. We proceed in the following way.

Let $\{\mathbf{x}_j\}_{j=1}^{n}$ be a set of uniformly distributed pairwise distinct interpolation points in $\Omega\cup\partial\Omega$. Note that $\{\mathbf{x}_j\}_{j=1}^{ni}\subseteq\Omega$, $\{\mathbf{x}_j\}_{j=ni+1}^{ni+nb1}\subseteq\partial\Omega_1$, and $\{\mathbf{x}_j\}_{j=ni+nb1+1}^{n}\subseteq\partial\Omega_2$, and $n=ni+nb1+nb2$. $\|\cdot\|$ denotes the Euclidean norm. Then we seek to approximate u by $\hat{u}$ as follows:

$$u(\mathbf{x})\simeq\hat{u}(\mathbf{x})=\sum_{i=1}^{n}a_j\varphi_\alpha(r_j),\quad \mathbf{x}\in\Omega\cup\partial\Omega, \tag{8}$$

where $r_j=\|\mathbf{x}-\mathbf{x}_j\|$, and $\varphi_\alpha(r_j)=\varphi(r_j/\alpha)$ is a CS-RBF with scaling factor α.

According to Kansa's method [29], the coefficient $\{a_j\}_{j=1}^{n}$ can be obtained by the collocation approach

$$\sum_{j=1}^{n}a_j\Delta\varphi_\alpha(\|\mathbf{x}_i-\mathbf{x}_j\|)=f(\mathbf{x}_i),\quad 1\le i\le ni, \tag{9}$$

$$\sum_{j=1}^{n}a_j\varphi_\alpha(\|\mathbf{x}_i-\mathbf{x}_j\|)=g(\mathbf{x}_i),\quad ni+1\le i\le ni+nb1, \tag{10}$$