Engineering Analysis with Boundary Elements 57 (2015) 23–36

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

# The localized method of approximated particular solutions for solving two-dimensional incompressible viscous flow field

C.Y. Lin [a,b], M.H. Gu [c], D.L. Young [a,b,*], J. Sladek [d], V. Sladek [d]

[a] Department of Civil Engineering, National Taiwan University, Taipei 10617, Taiwan
[b] Hydrotech Research Institute, National Taiwan University, Taipei 10617, Taiwan
[c] National Center for Research on Earthquake Engineering, Taipei 10668, Taiwan
[d] Institute of Construction and Architecture, Slovak Academy of Sciences, Bratislava 84503, Slovakia

ARTICLE INFO

*Article history:*
Received 1 May 2014
Received in revised form
9 November 2014
Accepted 23 November 2014
Available online 30 December 2014

*Keywords:*
Meshless
Localized method of approximated particular solutions
Incompressible flows
Navier–Stokes equations

ABSTRACT

The purpose of this paper is to demonstrate that the localized method of approximated particular solutions (LMAPS) is a stable, accurate tool for simulating two-dimensional incompressible viscous flow fields with Chorin's projection method. Totally there are two numerical experiments conducted: the two-dimensional lid-driven cavity flow problem, and the two-dimensional backward facing step problem. Throughout this study, the LMAPS has been tested by non-uniform point distribution, extremely narrow rectangular domain, internal flow, velocity or pressure driven flow and high velocity or pressure gradient, etc. All results are similar to results of finite element method (FEM) or other literature, and it is concluded that the LMAPS has high potential to be applied to more complicated engineering applications.



## 1. Introduction

The incompressible viscous flow is known for its applications in fields of hydraulics, liquid metal casting, blood circulation, etc. Incompressible viscous flow is used to describe flows with constant density within a fluid parcel. This does not imply, however, that the fluid itself must be incompressible; limited compressibility is acceptable for some compressible flow problems with suitable given conditions. The main advantage of assuming a flow to be incompressible is that it simplifies the equations that describe flow motion, which is known as the incompressible Navier–Stokes equations.

There are various algorithms developed to approximate the incompressible Navier–Stokes equations, the algorithms can be roughly categorized into coupled or decoupled schemes, where the decoupled methods make the equations for numerical implementation independent to each other by rearranging the system. The decoupled methods, more commonly used in computational fluid dynamics (CFD), includes methods such as Chorin's projection method [1,2], or the semi-implicit method for pressure-linked equations (SIMPLE) [3,4]. Another way to categorize the algorithms is to determine whether the algorithm is based on analyzing the primitive variables, velocity component and pressure, or not. Both of the earlier mentioned Chorin's project method or SIMPLE method used primitive variable formulation. For algorithms using vorticity or stream functions other than primitive variables, there are velocity–vorticity formulation [5,6], vorticity–stream function formulation [7], and others. The well-known Chorin's projection method is chosen to approximate the solutions of the incompressible Navier–Stokes equations by the LMAPS in this study.

The projection method solves the incompressible Navier–Stokes equations in three stages. The first stage is to split the operators into Burgers' equations to obtain the intermediate velocity numerically; the second stage is to take divergence into the split equations with pressure term to obtain the pressure Poisson's equation, and the third stage is to execute velocity correction with intermediate velocity and pressure obtained in the two former stages.

In CFD, such tasks are mostly executed via mesh-dependent numerical methods, including the finite difference method (FDM) [8], FEM [9], the finite volume method (FVM) [10], or the boundary element method (BEM) [11]. These methods have already been commonly used in scientific researches and engineering applications. However, the mesh-dependent numerical methods require mesh generation or numerical quadrature, which may be difficult to process during the numerical implementations, especially for multi-dimensional problems. In order to avoid mesh generation and numerical quadrature, various meshless numerical methods, such as the smoothed particle hydrodynamics (SPH) method [12], the multiquadrics (MQ) collocation

* Corresponding author at: Department of Civil Engineering, National Taiwan University, Taipei 10617, Taiwan.
*E-mail address:* dlyoung@ntu.edu.tw (D.L. Young).

http://dx.doi.org/10.1016/j.enganabound.2014.11.035
0955-7997/

***Symbol*: All variables used in this paper are formatted in italic or regular style, only the vectors and matrices are formatted in arrow or bold style. The arrow style stands for vector with variables of different space dimensions, and bold style indicates vectors with variables of different space points.**

*Subscripts*

| | |
|---|---|
| $d$ | $d$th dimension of space |
| $i$ | $i$th point of global domain |
| $j$ | $j$th point of global domain |
| $k$ | $k$th point of local domain |
| $l$ | $l$th point on a certain direction |
| $m$ | $m$th point of local domain |

method [13], the method of fundamental solutions (MFS) [14,15], the boundary knot method [16], the boundary particle method [17], the singular boundary method [18], the finite point method (FPM) [19], the local radial basis function-based differential quadrature method (LRBF-DQ) [20–22], the method of approximated particular solutions (MAPS) [23–25] and the meshless local Petrov–Galerkin method (MLPG) [26,27], have been developed.

Among the meshless methods, some, like the MQ or the MAPS, are categorized as the global-type meshless methods. Global-type methods require interpolation with all collocation points within the global domain, often causing the resultant interpolation matrix dense and ill-conditioning, which makes the calculation inefficient and unstable. In order to avert the above setbacks when dealing with large-scale computation problems, as recent literature shows, intense researches have been focused on various localization techniques for the global-type meshless methods [28,29].

The localization technique allows the numerical method to approximate the solution of given partial differential equations (PDEs) with less local influence points. By reformulating the dense resultant interpolation matrix into a sparser matrix, it reduces the risk of ill-conditioning and requires less computational time and memory loading. The localization techniques can be applied to the MQ and the MAPS, developing them into the localized multi-quadric method (LMQ) [28], the local radial basis functions differential quadrature method (LRBFDQ) [20], and the LMAPS [29].

The major concept of the localization for the MAPS was introduced by Yao et al. [29]. Instead of considering all points within the global domain, researchers choose a limited number of weighting points within a local influence area. Thus we can reformulate the algorithm to obtain the solutions directly instead of obtaining the weighting coefficients of the MAPS which are more physical oriented than the global formulation. For the LMAPS, this breakthrough maintains the advantage of meshless methods and improves from the MAPS to be more physical, robust and efficient.

The LMAPS has already been verified for solving many PDEs [30–32], it can be reasonably expected that the projection method should also be processed well by the LMAPS. The purpose of this paper is to validate the LMAPS as a stable, accurate tool for simulating the two-dimensional incompressible viscous flow field with the projection method. Two numerical experiments are conducted: the two-dimensional lid-driven cavity flow problem, and the two-dimensional backward facing step problem. The results in this paper are compared well with the results from FEM or literature data, which verifies the capability of the LMAPS.

The details of the LMAPS and its formulation of Chorin's projection method are explained in the following sections.

## 2. Governing equations and essential conditions

### 2.1. Governing equations

The computational domain $\Omega$ includes the boundary $\partial\Omega$. The dimensionless Navier–Stokes equations for the viscous incompressible flow field are listed as follows:

$$\nabla \cdot \vec{u} = 0 \tag{1}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2 \vec{u} \tag{2}$$

Here $\vec{u}(\vec{x},t) = (u_1, u_2)$ is the vector of velocity; $\vec{x} = (x_1, x_2) \in R^2$ is the vector of space; $t$ is time; $p$ is pressure; and Re is the Reynolds number.

### 2.2. Initial conditions and boundary conditions

In order to obtain unique solutions for the governing equations, the essential conditions such as the initial conditions and boundary conditions are required.

The initial condition is given in the general form as

$$\vec{u}(\vec{x},t)|_{t=0} = \vec{\sigma}(\vec{x}) \tag{3}$$

Dirichlet type and Neumann type boundary conditions are

$$\vec{u}(\vec{x},t)|_{\vec{x} \in \partial\Omega} = \vec{\beta}_{Dirichlet}(\vec{x}_b, t) \tag{4}$$

$$\frac{\partial \vec{u}(\vec{x},t)}{\partial n}\Big|_{\vec{x} \in \partial\Omega} = \vec{\beta}_{Neumann}(\vec{x}_b, t) \tag{5}$$

Here $\vec{\sigma}(\vec{x}) = (\sigma_1, \sigma_2)$ is the given initial condition; $\vec{\beta}(\vec{x}_b, t) = (\beta_1, \beta_2)$ is the given boundary condition; $\vec{x}_b \in \vec{x} \cap \partial\Omega$ is the space vector of boundary points; and $n$ is the unit outward normal direction of boundary.

## 3. Numerical methods

### 3.1. The projection method

Developed by Chorin in 1968 [1], the projection method is applied to approximate the solution of the incompressible Navier–Stokes equations. The method is based on splitting the operators of the momentum equations into three sequential stages to be processed separately. The derivation begins with the discretization of the time-domain by using the explicit Euler scheme and introduces the intermediate velocity.

$$\frac{\vec{u}^{n+1} - \vec{u}^{*} + \vec{u}^{*} - \vec{u}^{n}}{\Delta t} = -\left(\vec{u}^{n} \cdot \nabla\right)\vec{u}^{n} - \nabla p^{n+1} + \frac{1}{\mathrm{Re}}\nabla^2 \vec{u}^{n}, \tag{6}$$

where $\vec{u}^{*}$ depicts the intermediate velocity, and all superscripts denote the time steps.

Next, by operator splitting, the Navier–Stokes equations become the Burgers' equations which, provides the explicit Euler scheme to obtain the intermediate velocity $\vec{u}^{*}$, namely the first stage for the