Engineering Analysis with Boundary Elements 57 (2015) 66–71

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

A regularized multi-level technique for solving potential problems by the method of fundamental solutions

Csaba Gáspár

Széchenyi István University, Egyetem tér 1, H-9026 Győr, Hungary

ARTICLE INFO

Article history:
Received 6 October 2013
Accepted 7 May 2014
Available online 6 June 2014

Keywords:
Meshless method
Method of fundamental solutions
Regularization
Desingularization
Multi-level method

ABSTRACT

The method of fundamental solutions is investigated in the case when the source points are located along the boundary of the domain of the original problem and coincide with the collocation points. The appearing singularities are eliminated by several techniques: by using approximate but continuous fundamental solutions (regularization) and via auxiliary subproblems to avoid the stronger singularities that appear in the normal derivatives of the fundamental solution (desingularization). Both monopole and dipole formulations are investigated. A special iterative solution algorithm is presented, which converts the original (mixed) problem to a sequence of pure Dirichlet and pure Neumann subproblems. The pure subproblems can be handled efficiently by using conjugate gradients. The efficiency is significantly increased by embedding the resulting method in a natural multi-level context. At the same time, the problem of the use of highly ill-conditioned matrices is also avoided.

1. Introduction

Elliptic partial differential equations play an essential role in a lot of fields of application. Modelling stationary phenomena such as diffusion in fluids or in gases, heat transfer in machines e.g. in parts of traditional or hybrid cars, seepage through porous media lead to solving elliptic problems. The usual implicit time discretization techniques of time-dependent problems result in elliptic problems as well, at every time step.

To handle elliptic problems *in a meshless way*, the popular Boundary Element Method is not suitable, since it requires a boundary mesh structure. The strength of the meshless methods is to circumvent the generation of both domain and boundary mesh. So far, a number of boundary meshless methods have been developed e.g. the boundary knot method [3,4] which uses nonsingular general solutions, or the method of fundamental solutions (MFS, see e.g. [1]), which is based on the fundamental solution of the applied partial differential operator, i.e. on solutions with singularities. In this paper, we restrict ourselves to the MFS applied to homogeneous problem. (For non-homogeneous problems, the approach can be combined with the well-known principle of the Method of Particular Solutions).

Consider a second-order elliptic homogeneous linear partial differential equation:

$$Lu = 0 \quad \text{in } \Omega \tag{1}$$

defined in a sufficiently smooth domain Ω supplied with mixed boundary conditions:

$$u|_{\Gamma_D} = u_0, \quad \left.\frac{\partial u}{\partial n}\right|_{\Gamma_N} = v_0, \tag{2}$$

where $\Gamma := \partial\Omega$ denotes the boundary of Ω, which has a disjoint decomposition into a Dirichlet part Γ_D and a Neumann part Γ_N.

In its traditional form, the MFS produces an approximate solution of the problem (1) and (2) in the following form:

$$u_N(x) = \sum_{j=1}^{N} \alpha_j \Phi(x - \tilde{x}_j), \tag{3}$$

where Φ is a fundamental solution of the operator L, i.e. $\Delta\Phi = \delta$ (here δ denotes the Dirac distribution concentrated at the origin). The predefined points $\tilde{x}_1, \ldots, \tilde{x}_N$ (the *source points*) are located outside of the domain Ω. The a priori unknown coefficients $\alpha_1, \ldots, \alpha_N$ can be computed by enforcing the boundary conditions:

$$\begin{aligned} &\sum_{j=1}^{N} \alpha_j \Phi(x_k - \tilde{x}_j) = u_0(x_k) \quad (x_k \in \Gamma_D), \\ &\sum_{j=1}^{N} \alpha_j \frac{\partial \Phi}{\partial n_k}(x_k - \tilde{x}_j) = v_0(x_k) \quad (x_k \in \Gamma_N), \end{aligned} \tag{4}$$

where $x_1, \ldots, x_N \in \Gamma$ are predefined *boundary collocation points*. Then the function u_N exactly satisfies Eq. (1), and exhibits singularities at the source points.

Instead of the formulation (3) (called *monopole formulation* hereafter), it is often more advantageous to use the *dipole*

E-mail address: gasparcs@sze.hu

http://dx.doi.org/10.1016/j.enganabound.2014.05.002
0955-7997/

formulation, where the approximate solution of (1) and (2) is sought in the following form:

$$u_N(x) = \sum_{j=1}^{N} \alpha_j \frac{\partial \Phi}{\partial n_j}(x - \tilde{x}_j). \tag{5}$$

Again, Φ is the fundamental solution of the operator L, and the source points $\tilde{x}_1, \ldots, \tilde{x}_N$ are located outside of Ω. Now the coefficients $\alpha_1, \ldots, \alpha_N$ can be computed by solving the linear system:

$$\begin{aligned} &\sum_{j=1}^{N} \alpha_j \frac{\partial \Phi}{\partial n_j}(x_k - \tilde{x}_j) = u_0(x_k) \quad (x_k \in \Gamma_D), \\ &\sum_{j=1}^{N} \alpha_j \frac{\partial^2 \Phi}{\partial n_k \partial n_j}(x_k - \tilde{x}_j) = v_0(x_k) \quad (x_k \in \Gamma_N). \end{aligned} \tag{6}$$

Note that the monopole and the dipole formulations can be considered as meshless discretizations of the indirect BEM based on single layer and double layer potentials, respectively.

A common disadvantage of the traditional forms (3) and (5) is the use of external source points, the location of which can be hardly automatized.

Though the MFS has excellent accuracy properties (see e.g. [10] and references therein), the systems (4) and (6) are highly ill-conditioned in general, which is a severe drawback of the method. This is the case especially when the source points are located far from the boundary. On the other hand, if they are close to the boundary, the systems (4) and (6) become much better conditioned, however, the accuracy goes wrong due to the appearance of numerical singularities at the boundary collocation points.

A usual technique is to define the source and the boundary collocation points to coincide. Special techniques are required to avoid the problem of singularity (regularization and desingularization, see e.g. [14,7,12,8]).

In this paper, we investigate some regularized versions of both the monopole and the dipole formulations. It turns out that, from a computational point of view, the dipole formulation is much more advantageous for handling *pure Dirichlet problems*, while for *pure Neumann problems*, the monopole formulation overperforms the dipole formulation. For mixed boundary conditions, a special iterative technique is proposed which converts the original mixed problem to a convergent sequence of pure Dirichlet and pure Neumann subproblems. This results in a computationally efficient method, which avoids also the problem of highly ill-conditioned linear systems. The efficiency can be increased further by embedding the method in a natural multi-level context.

2. Regularization and desingularization

In the rest of the paper, suppose that the source points and the boundary collocation points $x_1, \ldots, x_N \in \Gamma$ coincide. Then the monopole formulation has the form

$$u_N(x) = \sum_{j=1}^{N} \alpha_j \Phi(x - x_j), \tag{7}$$

where the coefficients $\alpha_1, \ldots, \alpha_N$ can be computed by solving the algebraic system:

$$\begin{aligned} &\sum_{j=1}^{N} \alpha_j A_{kj} = u_0(x_k) \quad (x_k \in \Gamma_D), \\ &\sum_{j=1}^{N} \alpha_j B_{kj} = v_0(x_k) \quad (x_k \in \Gamma_N). \end{aligned} \tag{8}$$

Similarly, the dipole formulation has the form

$$u_N(x) = \sum_{j=1}^{N} \alpha_j \frac{\partial \Phi}{\partial n_j}(x - x_j), \tag{9}$$

where the coefficients $\alpha_1, \ldots, \alpha_N$ solve the algebraic system:

$$\begin{aligned} &\sum_{j=1}^{N} \alpha_j C_{kj} = u_0(x_k) \quad (x_k \in \Gamma_D), \\ &\sum_{j=1}^{N} \alpha_j Q_{kj} = v_0(x_k) \quad (x_k \in \Gamma_N). \end{aligned} \tag{10}$$

Here the entries of the matrices A, B, C, Q are defined as follows:

$$\begin{aligned} &A_{kj} = \Phi(x_k - x_j), \quad B_{kj} = \frac{\partial \Phi}{\partial n_k}(x_k - x_j), \\ &C_{kj} = \frac{\partial \Phi}{\partial n_j}(x_k - x_j), \quad Q_{kj} = \frac{\partial^2 \Phi}{\partial n_k \partial n_j}(x_k - x_j) \end{aligned} \tag{11}$$

for $j \neq k$. Due to the singularity of the fundamental solution at the origin, the diagonal entries cannot be computed by the above definition.

For the proper definition of A_{kk}, one should replace the fundamental solution Φ with an approximate fundamental solution $\overline{\Phi}$, which has no singularity at the origin. Such an approximate fundamental solution can be defined e.g. by truncation. In polar coordinates:

$$\overline{\Phi}(r) := \begin{cases} \Phi(r) & \text{if } r \geq \dfrac{1}{c} \\ \Phi\left(\dfrac{1}{c}\right) & \text{if } r < \dfrac{1}{c}, \end{cases} \tag{12}$$

provided that Φ is a radial function i.e. it depends only on $r = ||x||$, which is often the case. Here c denotes a carefully chosen scaling constant which should remain inversely proportional to the characteristic distance of the boundary collocation points, when N varies. Another regularization technique is to replace Φ with the fundamental solution of the singularly perturbed fourth-order operator $L(I - (1/c^2)L)$, where I denotes the identity operator and c is again a scaling constant, see [8] for details. Thus, the diagonal terms A_{kk} can be computed without difficulty. Using the simplest truncation, $A_{kk} = \overline{\Phi}(0)$, while for $j \neq k$, $A_{kj} = \Phi(x_k - x_j)$.

The proper definition of the diagonal terms B_{kk} is somewhat more difficult since the derivatives of Φ have stronger singularities at the origin than Φ itself. Let w be a smooth, easily computable particular solution of Eq. (1), then w can be approximated by the monopole formulation:

$$w_N(x) := \sum_{j=1}^{N} \beta_j \overline{\Phi}(x - x_j),$$

from where the coefficients $\beta_1, \ldots, \beta_N$ can be computed by solving (1) supplied with a pure Dirichlet condition. Computing the normal derivative of the particular solution w, we have the following:

$$\frac{\partial w_N}{\partial n}(x) = \sum_{j=1}^{N} \beta_j \frac{\partial \overline{\Phi}}{\partial n}(x - x_j).$$

Hence

$$\sum_{j \neq k} \beta_j B_{kj} + \beta_k B_{kk} = \frac{\partial w}{\partial n_k}(x_k) \quad (k = 1, 2, \ldots, N).$$

Thus, the diagonal terms B_{kk} can be defined as

$$B_{kk} := \frac{1}{\beta_k}\left(- \sum_{j \neq k} \beta_j B_{kj} + \frac{\partial w}{\partial n_k}(x_k) \right).$$

This is the *desingularization idea*, see e.g. [14,11,5] for details. Note that in the simplest case of the Laplace equation w can often be chosen e.g. by $w := \equiv 1$; in the case of the modified 2D Helmholtz equation $\Delta u - \lambda^2 u = 0$, a particular solution $w(x) := I_0(\lambda \cdot ||x||)$ can be used, where I_0 denotes the familiar modified Bessel function of the first kind, and so on. By a proper choice of the particular solution w, one can ensure that the coefficients β_k do not vanish, so that the