Engineering Analysis with Boundary Elements 53 (2015) 9–17

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Complex variables-based approach for analytical evaluation of boundary integral representations of three-dimensional acoustic scattering

Fatemeh Pourahmadian, Sofia G. Mogilevskaya *

Department of Civil, Environmental & Geo-Engineering, University of Minnesota, Minneapolis, United States

ARTICLE INFO

Article history:
Received 21 July 2014
Received in revised form
29 August 2014
Accepted 19 November 2014
Available online 31 December 2014

Keywords:
Acoustics
Helmholtz equation
BEM
Analytical integration
Complex variables

ABSTRACT

The paper presents the complex variables-based approach for analytical evaluation of three-dimensional integrals involved in boundary integral representations (potentials) for the Helmholtz equation. The boundary element is assumed to be planar bounded by an arbitrary number of straight lines and/or circular arcs. The integrals are re-written in local (element) coordinates, while in-plane components of the fields are described in terms of certain complex combinations. The use of Cauchy–Pompeiu formula (a particular case of Bochner–Martinelli formula) allows for the reduction of surface integrals over the element to the line integrals over its boundary. By considering the requirement of the minimum number of elements per wavelength and using an asymptotic analysis, analytical expressions for the line integrals are obtained for various density functions. A comparative study of numerical and analytical integration for particular integrals over two types of elements is performed.

1. Introduction

This paper extends the complex variables-based integration technique recently developed for three-dimensional potential and elastostatic problems [14] to three-dimensional acoustic scattering problems described by the Helmholtz equation in frequency domain. It is well-known that the solutions of the latter problems can be represented by certain integrals, or combinations of integrals, over the boundary of the domain of interest, see e.g. [1,5,7]. The unknown fields in such representations can be found by solving the so-called boundary integral equations. The Boundary Element Method (BEM), see e.g. [1,5,9], is a numerical technique for solving these equations. The technique leads to the discretized equations that involve the integrals over the boundary elements used to approximate the boundaries of the simulation domains. Analytical evaluation of the integrals is an attractive option since it leads to higher accuracy of the computation and to the reduction of its cost. This may also facilitate the use of fast methods [11,20] and can be utilized (along with other methods such as cubature method and nonlinear regularizing transformations, e.g. see [16]) to form a robust framework for evaluation of BEM integrals in a more general context.

Closed-form results for the integrals involved in integral representations of the potential and elasticity theories are reported in many publications, especially for two-dimensional problems with straight elements [3,10,13,22] and for three-dimensional problems with triangular and rectangular elements [2,12,15,17,18,21,23,24]. However, only few papers report analytical results for the BEM integrals in acoustic scattering. One of such papers [8] presents a semi-analytical approach to evaluate singular and near singular double integrals involved in Galerkin formulation for the Helmholtz equation. The method employs constant approximations for the basic functions and uses triangular boundary elements (in coplanar or parallel planes). Analytical expressions for these integrals are provided for the singular parts of the Helmholtz Green's functions that coincide with the kernels of a single- and double-layer potentials of the Laplace equation, while numerical integration is used for the remaining dynamic part. The method is based on an integration formula for homogeneous functions that reduce an integral over an *N*-dimensional domain into an integral over its boundary.

The analytical approach presented in [25] (for 3D wave propagation) and in [26] (for transient heat conduction) also deals with singular and hypersingular BEM integrals over planar elements. The approach employs rectangular elements and constant approximations for the unknowns and relies on the Fourier series representation for the Helmholtz fundamental solution. It is shown that the method leads to satisfactory results, however, the minimum required number of terms in the Fourier expansion may become large or

* Corresponding author. Tel.: +1 6126254810.
E-mail address: mogil003@umn.edu (S.G. Mogilevskaya).

http://dx.doi.org/10.1016/j.enganabound.2014.11.019
0955-7997/

sensitive to some specific parameters. Also, the possibility of spatial contamination can be another drawback of the method.

Another somewhat relevant paper [28] reports analytical expressions for moment integrals in the diagonal fast multipole BEM to solve 3-D acoustic wave problems. The paper also employs constant approximation for the unknowns and triangular elements.

In the present paper we use the complex variables-based technique proposed in [14] to evaluate three-dimensional integrals in the BEM formulations related to the Helmholtz equation. The technique is based on the complex integral representations that reduce the area integrals to those over the element contour. To use these representations, various complex combinations of in-plane fields and geometrical parameters are formed. For polynomial approximations of density functions in the BEM formulations, the procedure allows for analytical integrations of all integrals (regular, singular, and hypersingular) over planar elements bounded not only by straight lines but also by circular arcs (and, possibly, by other simple curves).

The structure of the paper is as follows. In Section 2, we present real variables-based integral representations involved in typical BEM formulations for the Helmholtz equation. In Section 3, we review various complex notations for geometry and fields and introduce generic complex integral. In Section 4, this integral is reduced to a contour integral using Cauchy–Pompeiu integral representation. In Section 5, the closed form expressions for this integral over a straight segment and a circular arc are presented. In Section 6, comparative analyses of numerical and analytical integration for particular integrals over elements of two types are performed. The outcome of the present study is summarized in Section 7 and its implications are discussed.

2. Integral representations of acoustic scattering in $\mathbb{R}^3$

The time-harmonic scalar wave propagation is governed by the following Helmholtz equation:

$$\Delta u + k^2 u = 0, \quad u = u(\boldsymbol{x}, \omega) = \mathrm{Re}[u(\boldsymbol{x}) e^{-\mathrm{i}\omega t}], \quad k = \omega / c, \tag{1}$$

where u is the scalar field variable that is a function of position $\boldsymbol{x} \in \mathbb{R}^3$ and frequency ω, k is the wave number, and c is the medium's sound speed. The typical boundary element method formulations in acoustics involve the following integrals over the boundary, e.g. [1–3]:

- Single-layer potential

$$\int_S \frac{1}{4\pi r} v(\boldsymbol{\zeta}) e^{ikr} \, \mathrm{d}S_{\zeta}, \tag{2}$$

- Double-layer potential

$$\int_S w(\boldsymbol{\zeta}) \frac{\partial}{\partial \mathbf{n}(\boldsymbol{\zeta})} \left[\frac{1}{4\pi r} e^{ikr} \right] \mathrm{d}S_{\zeta}, \tag{3}$$

- Adjoint double-layer potential

$$\frac{\partial}{\partial \mathbf{n}(\boldsymbol{x})} \int_S \frac{1}{4\pi r} v(\boldsymbol{\zeta}) e^{ikr} \, \mathrm{d}S_{\zeta}, \tag{4}$$

- Hypersingular potential

$$\frac{\partial}{\partial \mathbf{n}(\boldsymbol{x})} \int_S w(\boldsymbol{\zeta}) \frac{\partial}{\partial \mathbf{n}(\boldsymbol{\zeta})} \left[\frac{1}{4\pi r} e^{ikr} \right] \mathrm{d}S_{\zeta}, \tag{5}$$

where $r = |\boldsymbol{\zeta} - \boldsymbol{x}|$ is the distance between the boundary point $\boldsymbol{\zeta} \in S$ and the field point $\boldsymbol{x} \in \mathbb{R}^3$; $\mathbf{n}(\boldsymbol{\zeta})$ denotes the unit normal vector to the boundary at the point $\boldsymbol{\zeta}$, while $\mathbf{n}(\boldsymbol{x})$ is the normal vector to some plane containing the point $\boldsymbol{x}$; the two scalars $v(\boldsymbol{\zeta})$ and $w(\boldsymbol{\zeta})$ are the so-called density functions. Eq. (1) is automatically satisfied when u is described by one of the expressions of Eqs. (2)–(5), or their linear combinations.

3. Generic integral involved in potentials (2)–(5)

With reference to Fig. 1, S is a planar boundary element consisting of a regular domain bounded by a piece-wise smooth and oriented contour ∂S that does not intersect itself. The element (local) coordinates are indicated by $(\zeta_1, \zeta_2, \zeta_3)$ so that ζ_3 is directed along the normal vector $-\mathbf{n}(\boldsymbol{\zeta})$, whereas ζ_1 and ζ_2 are in-plane directions chosen in such a way that $(\zeta_1, \zeta_2, \zeta_3)$ is a right handed coordinate system. Furthermore, assume that z is the projection of the field point $\boldsymbol{x}$ onto the element's plane. It should be mentioned that direction of travel on ∂S is assumed to be counter-clockwise.

As in [14], we employ the following complex combinations:

$$z = x_1 + \mathrm{i}x_2, \quad \bar{z} = x_1 - \mathrm{i}x_2,$$
$$\tau = \zeta_1 + \mathrm{i}\zeta_2, \quad h = \zeta_3 - x_3, \tag{6}$$

wherein z, $\bar{z}$ – also τ, $\bar{\tau}$ – are hereafter treated as independent variables. Using these combinations, the distance r is expressed as follows:

$$r = \sqrt{(\tau - z)(\bar{\tau} - \bar{z}) + h^2}. \tag{7}$$

In the following we would also use the Wirtinger calculus,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left[\frac{\partial}{\partial x_1} - \mathrm{i}\frac{\partial}{\partial x_2}\right], \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial}{\partial x_1} + \mathrm{i}\frac{\partial}{\partial x_2}\right]. \tag{8}$$

The use of Eqs. (6)–(8) and the chain differentiation rule leads to the following useful interrelations:

$$r_{,1} = \frac{\partial r}{\partial z} + \frac{\partial r}{\partial \bar{z}}, \quad r_{,2} = \mathrm{i}\left(\frac{\partial r}{\partial z} - \frac{\partial r}{\partial \bar{z}}\right), \quad r_{,3} = -\frac{\partial r}{\partial h}, \tag{9}$$

where $r_{,j} = \partial r / \partial x_j$.

In this setting, polynomial approximations of the density functions $v(\boldsymbol{\zeta})$, $w(\boldsymbol{\zeta})$ result in linear combinations of the terms $(\tau - z)^m (\bar{\tau} - \bar{z})^n$, e.g. the monomial $(\zeta_1)^2$ transforms to the following

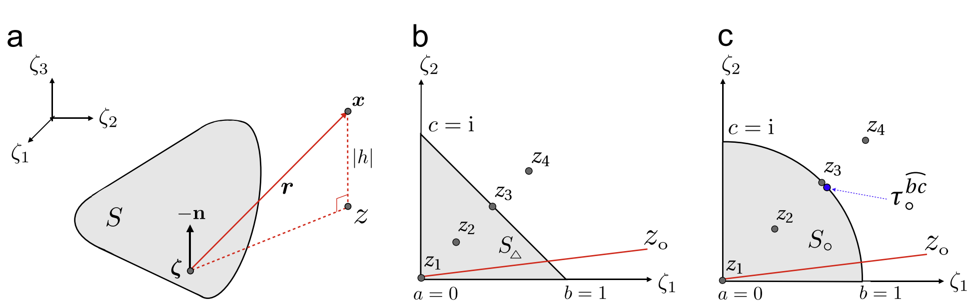

Fig. 1. Planar boundary elements: (a) typical element, (b) triangular element, (c) circular sector.