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Meshless simulation of stochastic advection–diffusion equations based on radial basis functions

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ABSTRACT

In this paper, a numerical technique is proposed for solving the stochastic advection–diffusion equations. Firstly, using the finite difference scheme, we transform the stochastic advection–diffusion equations into elliptic stochastic partial differential equations (SPDEs). Then the method of radial basis functions (RBFs) based on pseudospectral (PS) approach has been used to approximate the resulting elliptic SPDEs. In this study, we have used generalized inverse multiquadrics (GIMQ) RBFs, to approximate functions in the presented method. The main advantage of the proposed method over traditional numerical approaches is directly simulating the noise terms at the collocation points in each time step. To confirm the accuracy of the new approach and to show the performance of the selected RBFs, four examples are presented in one, two and three dimensions in regular and irregular domains. For test problems the statistical moments such as mean, variance and standard deviation are computed.

1. Introduction

Partial differential equations (PDEs) have been widely used to model many problems in applied sciences and engineering. PDEs have been extensively studied in the literature and several numerical methods have been developed for finding their approximate solutions. In many practical situations such ideal information is rarely encountered. For example, this occurs in advection–diffusion models arising in ground water flows where exact knowledge of the permeability of the soil, magnitude of source terms, inflow or outflow conditions are exactly not known. The existence of uncertainties in such problems can be described by random fields. This requires to include, in the PDEs modeling, a rational assessment of uncertainty. Consequently, this leads to the notion of stochastic PDEs (SPDEs) [2–4,12,16,48].

The numerical solution of SPDEs becomes a fast growing research area. The finite element methods (FEMs) use polynomial interpolation functions to approximate the terms in the equations over small parts of domains, called elements. By assembling all influence matrices which express the properties of each element, a global matrix is obtained. Similar to deterministic case the FEMs use weak form to approximate the governing stochastic differential equation. FEMs have been applied for the numerical solution of the stochastic boundary and initial problems successfully, for instance see [1,9]. Finite difference methods (FDMs) are another tools that have been applied for the numerical solution of SPDEs, for instance see [1,42,43]. FDMs approximate the governing deterministic differential equation using truncated Taylor series expansion, which results in a system of algebraic equations. The method of Wiener chaos expansions is another technique that has been applied for solving SPDEs, see for example [34]. As mentioned in [41], in this approach, random fields are discretized using polynomial chaos resulting in a set of coupled deterministic boundary-value problems to be solved. However, the Wiener chaos expansions have some limitations in application to stochastic boundary-value problems with complex stochastic forcing terms. For instance, large number of chaos coefficients in expansions are needed to accurately compute small scales. In addition, many realizations have to be performed to obtain accurate estimates of the required statistical characteristics. Therefore, Wiener chaos expansions are computationally expensive.

The stochastic spectral collocation method [37] and the Itô Taylor expansions method [36] are other numerical tools that were discussed for solving SPDEs.

Meshless methods are powerful numerical tools that have been applied for solving many problems in engineering and applied mathematics. These methods do not require a mesh to discretize the domain of the problem under consideration and the approximate solution is constructed entirely based on a set of scattered nodes. Several domain type meshless methods such as smooth particle hydrodynamics method, element free Galerkin method, reproducing kernel particle method, the point interpolation

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method, the moving least squares method, the meshless Petrov–Galerkin method have been proposed and achieved remarkable progress in solving a wide range of applied mathematic and engineering problems, for example see [19,24,38,39,44–46].

The method of RBFs is one of the most recently developed meshless techniques that has attracted attention in engineering problems, for instance see [10,11,33,47,50]. Due to its simplicity to implement, it represents an attractive alternative to the domain decomposition methods like FDMs, FEMs as a solution method of nonlinear differential equations, for instance see [21–23,25,26].

RBFs are widely used for solving problems arising in financial mathematics, for instance see [5–7]. However, it is only since rather recently that the meshless method has been used to approximate solutions for SPDEs. Meshless method of RBFs has been applied for the numerical solution of time-dependent [14,40] and time-independent [29] SPDEs.

1.1. The main idea of the paper and literature review

Let $\mathcal{D} \subset \mathbb{R}^d, d=1,2,3$ be a regular open bounded domain in $\mathbb{R}^d$, which has a smooth boundary $\partial\mathcal{D}$. Also let $\mathcal{H}=L^2(\mathcal{D})$ denote a separable Hilbert space of functions defined on $\mathcal{D}$. Let $W(x,t)$, for $x \in \mathbb{R}^d$, $t \ge 0$, be a continuous Wiener process in a complete probability space $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ with a filtration $\{\mathcal{F}\}_{t=0}^{\infty}$ [16]. It has known mean and covariance kernel $q(x,y)$ as follows:

$$\mathbb{E}(W(x,t))=0,$$
$$\mathbb{E}(W(x,t)W(y,s))=\min\{t,s\}q(x,y), \quad x,y \in \mathbb{R}^d, \ \ 0 \le s,t \le T. \tag{1}$$

Our concern in the current work is developing RBFs based on pseudospectral (PS) method for solving the stochastic advection–diffusion equations of the following form [14,41]:

$$\begin{cases} du+(\nu\nabla \cdot u-\gamma\nabla^2 u-f)\,dt=\sigma\,dW(t), & \\ u(x,0)=u_0 \in \mathcal{H}, & x \in \mathcal{D}, \\ u(x,t)|_{\partial\mathcal{D}}=g, & t \in (0,T), \end{cases} \tag{2}$$

where ∇ denotes the gradient operator and $\sigma>0$. In addition, $\nu>0$ and $\gamma>0$ are considered to be positive constants quantifying the advection and diffusion processes, respectively. In addition, the functions f and g are given such that problem (2) has a unique solution [12,13]. We have used an additive noise as random forcing in the stochastic advection–diffusion equations. However, the method presented in this paper can be implemented for solving more general additive noise acting as forcing terms in the PDEs. The function $u(x,t)$ represents, for example, the temperature in heat equations or concentration in advection–diffusion equations [41]. In addition, model (2) has been used to describe heat transfer in a draining film, water transfer in soils, dispersion of tracers in porous media, the intrusion of salt water into fresh water aquifers, the spread of pollutants in rivers and streams, the dispersion of dissolved material in estuaries and coastal seas, contaminant dispersion in shallow lakes, the absorption of chemicals into beds, the spread of solute in a liquid flowing through a tube, long-range transport of pollutants in the atmosphere. For more descriptions see [17].

The authors of [41] provided the method of lines for solving stochastic advection–diffusion equations in one and two dimensions. Wan et al. [49] solved stochastic advection–diffusion equations using Wiener chaos expansions. In addition, Shardlow [42,43], Gyöngy [30], Davie [15] proposed the FDM, Yan [51], Barth and Lang [8] used FEM, Allen et al. [1], Du and Zhang [27] used FDM and FEM, authors of [31,32] used eigenfunction, FDMs and wavelets for approximating the solution of parabolic stochastic PDEs.

In this paper, we will concentrate on the numerical solution of the stochastic advection–diffusion equations in one, two and three dimensions using RBFs based on PS method as a truly meshless method.

The layout of the rest of this paper is as follows: in the first part of Section 2, we present the temporal discretization via FDM. The spatial discretization with the RBFs based PS method is explored in the other part of Section 2. The numerical simulations of stochastic advection–diffusion equations in one, two and three dimensions are presented in Section 3 which show the satisfactory performance of the presented method. The last section concludes a brief conclusion.

2. Mathematical formulation

In this section, we formulate RBFs based on PS method for solving Eq. (2). Here the time splitting approach has been employed to transform (2) into elliptic SPDE then the method of RBFs based on PS idea has been applied for solving the resulting elliptic SPDE.

2.1. Temporal discretization

In the current work, we employ a time-stepping scheme to overcome the time derivative. For this purpose let us discretize (2) in time by the implicit Euler scheme at equally spaced time points

$$0=t_0 \le t_1 \le \cdots \le t_n=T,$$

so we arrive at

$$u^n-u^{n-1}+(\nu\nabla \cdot u^n-\gamma\nabla^2 u^n-f^n)\tau=\sigma\delta W_n, \tag{3}$$

where $u^n{:=}u(x,t_n)$, $\tau{:=}t_n-t_{n-1}$ and $\delta W_n{:=}W_{t_n}-W_{t_{n-1}}$. Let $\xi_x=\sigma\delta W_n$. Here ξ_x has the following mean and covariance function:

$$\mathbb{E}(\xi_x)=0,$$
$$\mathbb{E}(\xi_x\xi_y)=\sigma^2\tau q(x,y), \quad x,y \in \mathbb{R}^d.$$

So Eq. (3) with the corresponding boundary conditions becomes an elliptic SPDE of the form

$$\begin{cases} \mathcal{L}u=F+\xi_x & \text{in } \mathcal{D}, \\ u|_{\partial\mathcal{D}}=G, & \end{cases} \tag{4}$$

where $\mathcal{L}[\cdot]=[\cdot]+\tau\nu\nabla\cdot[\cdot]-\tau\gamma\nabla^2[\cdot]$, $u{:=}u^n$ is unknown, $F{:=}u^{n-1}+\tau f^n$, $G=:g^n$ and ξ_x are given known parts. Since it follows from (3) and the definition of Brownian motion that the noise increment $\sigma\delta W_n$ at each time instance t_n is independent of the solution u^{n-1} at the previous step, we simulate the Gaussian field with covariance structure $\sigma^2\tau q(x,y)$ at a finite collection of predetermined collocation points [14].

In the forthcoming subsection we present the RBFs method based on PS collocation idea for the numerical solution of the elliptic SPDE (4).

2.2. Spatial discretization

First, consider a finite collection of predetermined pairwise distinct collocation points

$$X_{\mathcal{D}}=\{x_1,x_2,\ldots,x_N\}\subset\mathcal{D}, \quad X_{\partial\mathcal{D}}=\{x_{N+1},\ldots,x_{N+M}\}\subset\partial\mathcal{D}.$$

Following [20,40] the solution of Eq. (4) can be written as expansion

$$u(x)=\sum_{j=1}^{N} c_j\mathcal{L}_2\phi(x,x_j)+\sum_{j=1}^{M} c_{j+N}\phi(x,x_j), \tag{5}$$

where $\phi_j{:=}\phi(x,x_j)$ are strictly positive definite radial functions, i.e., $\phi_j=\Phi(\|x-x_j\|)$ where $\|x-x_j\|$ denotes the distance between x and the jth nodal point x_j and $c_j, j=1,\ldots,N+M$ are coefficients to be determined. Also $\mathcal{L}_2\phi(x,x_j)$ means that we differentiate with respect to the second variable followed by evaluation at x_j. In this study, we will use the generalized inverse multiquadrics (GIMQ)