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Reconstruction of inaccessible boundary value in a sideways parabolic problem with variable coefficients—Forward collocation with finite integration method

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ABSTRACT

We investigate a sideways problem of reconstructing an inaccessible boundary value for parabolic equation with variable coefficients. Formulating the sideways problem into a sequence of well-posed direct problems (DP) and a system of Ordinary Differential Equations (ODE), we combine the recently developed finite integration method (FIM) with radial basis functions (RBF) to iteratively obtain the solution of each DP by solving an ill-posed linear system. The use of numerical integration instead of finite quotient formula in FIM completely avoids the well known roundoff-discretization errors problem in finite difference method and the use of RBF as forward collocation method (FCM) gives a truly meshless computational scheme. For tackling the ill-posedness of the sideways problem, we adapt the traditional Tikhonov regularization technique to obtain stable solution to the system of ODEs. Convergence analysis is then derived and error estimate shows that the error tends to zero when perturbation $\delta \to 0$. We can then obtain highly accurate and stable solution under some assumptions. Numerical results validate the feasibility and effectiveness of the proposed numerical algorithms.

1. Introduction

Boundary value determination problems arise from the textiles manufacture and design industry in which the surface heat and moisture transfer are to be determined. Consider the following parabolic equation which models the temperature on both sides of a thick wall or the temperature and/or humidity on an inaccessible surface of a body:

$$\begin{cases} u_t = \nabla \cdot (\kappa \nabla u) + \Theta(x,t), \quad x \in \Omega, \ t \in (0,T), \\ u(x,0) = v(x), \quad x \in \overline{\Omega}, \\ u|_{\Gamma_1} = w_1(x,t), \quad \left.\dfrac{\partial u}{\partial n}\right|_{\Gamma_2} = w_2(x,t), \ x \in \overline{\Omega}, \ t \in [0,T], \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, Γ_i, $i=1,2$, are part of the boundary $\partial\Omega$ and T is a prescribed number. The sideways parabolic problem is to recover the boundary value on the remaining part $\partial\Omega \backslash (\Gamma_1 \cup \Gamma_2)$.

For example, the heat and moisture transfer within textiles gives the following 1-D sideways parabolic problem:

$$\begin{cases} \varepsilon \dfrac{\partial C_a}{\partial t} = \dfrac{D_a \varepsilon}{\tau} \dfrac{\partial^2 C_a}{\partial x^2} - \gamma(x,t), \quad (x,t) \in (0,L) \times (0,T), \\ C_a(x,0) = C_0(x), \quad x \in [0,L], \\ C_a(L,t) = C_1(t), \quad \left.\dfrac{D_a \varepsilon}{\tau} \dfrac{\partial C_a}{\partial x}\right|_{x=L} = \dfrac{C_E(t)}{-C_a|}_{x=L} w_1 + (1/h_c), \quad t \in [0,T]. \end{cases} \tag{2}$$

See [1–6] for reference. Here, we intend to recover the boundary value $C_a(0,t)$, $t \in [0,T]$. For 2-D case, the sideways parabolic problem can be formulated as

$$\begin{cases} u_t = \nabla \cdot (\kappa \nabla u) + \Theta(x,y,t), \quad (x,y) \in \Omega, \ t \in (0,T), \\ u(x,y,0) = v(x,y), \quad (x,y) \in \overline{\Omega}, \\ u(x,c,t) = \varphi_1(x,t), u(x,d,t) = \varphi_2(x,t), \quad x \in [a,b], t \in [0,T], \\ u(b,y,t) = \psi_2(y,t), \quad u_x(b,y,t) = \chi(y,t), \ y \in [c,d], t \in [0,T], \end{cases}$$

where $\Omega = (a,b) \times (c,d)$. The inverse problem is then to recover the boundary value $u(a,y,t)$. In the following sections, we focus on solving the sideways problem for 1-D and 2-D cases. We note here that the proposed method is readily extendable to handle n-variate problems $(n \geq 3)$.

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For simplicity, we rewrite (2) in a unified form: Determine the boundary value $u(0,t)$ such that

$$\begin{cases} \frac{\partial u}{\partial t} = \alpha_1(x,t)\frac{\partial^2 u}{\partial x^2} + \alpha_2(x,t)\frac{\partial u}{\partial x} + \alpha_3(x,t)u + \Theta(x,t), & (x,t) \in (0,L) \times (0,T), \\ u(x,0) = v(x), \quad x \in [0,L], \\ u(L,t) = \psi(t), \quad u_x|_{x=L} = \chi(t), \ t \in [0,T]. \end{cases} \tag{3}$$

The corresponding direct problem (DP) is given by

$$\begin{cases} \frac{\partial u}{\partial t} = \alpha_1(x,t)\frac{\partial^2 u}{\partial x^2} + \alpha_2(x,t)\frac{\partial u}{\partial x} + \alpha_3(x,t)u + \Theta(x,t), & (x,t) \in (0,L) \times (0,T), \\ u(x,0) = v(x), \quad x \in [0,L], \\ u(0,t) = \varphi(t), \quad u(L,t) = \psi(t), \ t \in [0,T]. \end{cases} \tag{4}$$

If the left boundary value $\varphi(t)$ is unknown and the solution of (4) is denoted by $u[\varphi]$, then the 1-D sideways problem can be stated as follows: *Find a suitable $\varphi(t)$ such that $u[\varphi]$ is the solution of* (3).

It is well known that sideways problem for Laplace equation or heat equation is ill-posed in Hadamard's sense: any small change in the input data may result in a dramatic change in the solution. Some kinds of regularization techniques are necessary for stabilizing the computation. A number of numerical methods for the sideways parabolic problem have been proposed, which can be divided into two categories: finite difference method and Fourier transform method.

In [7], Murio developed a mollification method for solving a similar 1-D sideways parabolic problem. The method filters the noisy data by discrete convolution with a suitable averaging kernel and adopts explicit finite differences, marching in space, to numerically solve the associated well-posed problem. Murio [8] further improved the mollification method by incorporating the necessary initial filtering procedure into the marching scheme itself. From the regularization and approximation properties of a few time marching schemes, Eldén [9] demonstrated that time discretization prevents high frequencies in the solution from blowing up. In other words, finite difference in time has a regularizing effect.

The Fourier transform method was firstly used by Hào in [10] to solve the following 1-D sideways parabolic problem

$$\begin{cases} u_t = a(x)u_{xx} + b(x)u_x + c(x)u, & x > 0, t > 0, \\ u(x,0) = 0, \quad x \geq 0, \\ u(1,t) = g(t), \quad t \geq 0, \\ u(x, \ \cdot \ )|_{x \to \infty} \quad \text{is bounded,} \end{cases} \tag{5}$$

in which a stability estimate of the Hölder type for the solution was established for a stable solution by using the mollification method. The common technique adopted in Fourier transform method is to transform the original partial differential equation (PDE) into a system of Ordinary Different Equations (ODE) in Fourier domain. To stabilize the solution process in solving this system of ODEs, we usually eliminate high frequencies from the Fourier inversion formula to determine the cut-off by various regularization methods, which has been considered in [11–13]. In fact, the solution can be presented by

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi t} \frac{v(x,\xi)}{v(1,\xi)} \hat{g}(\xi)\, d\xi \tag{6}$$

and $v(x,\xi)$ is the solution of the following boundary value problem:

$$\begin{cases} i\xi v(x,\xi) = a(x)v_{xx} + b(x)v_x + c(x)v, \\ v(0,\xi) = 1, \\ \lim\limits_{x\to\infty} v(x,\xi) = 0, \quad \xi \neq 0. \end{cases} \tag{7}$$

For $\xi = 0$, it requires that $v(x,0)$ is bounded as x tends to ∞. The Fourier regularization method replaces the solution by

$$u^{\delta}_{\max}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi t} \frac{v(x,\xi)}{v(1,\xi)} \hat{g}(\xi)\chi_{\max}\, d\xi \tag{8}$$

where $\chi_{\max}$ is the characteristic function of interval $[-\xi_{\max}, \xi_{\max}]$ and $\xi_{\max}$ plays a role of regularization parameter. Xiong et al. in [14] proposed three spectral regularization methods to solve the general sideways parabolic equation. Other methods, such as Tikhonov method, wavelet and wave-Galerkin method, optimal approximations and optimal filtering method, can be found in [15–18]. Deng et al. in [19] introduced a new class of iteration methods to solve the inverse problem and proved that their methods are of order optimal under both *a priori* and *a posteriori* stopping rules.

Since the Fourier transform requires only the coefficient function in x, the cut-off can be used to avoid the tedious convolution process. In addition, if we append an additional term $f(x,t)$ or $f(u)$ to the right-hand side of the equation, then the Fourier transform may become invalid, especially for nonlinear $f(u)$.

For numerical approximation, we combine in this paper the recently developed finite integration method (FIM) with radial basis functions (RBF) [23–26] to iteratively obtain the solution of each DP by solving an ill-posed linear system. To tackle the ill-posedness of the sideways problem, we adapt the traditional Tikhonov regularization technique to obtain stable solution to the system of ODEs. The use of the RBF as forward collocation method (FCM) with the regularization technique gives a truly meshless computational scheme to cope with the ill-posedness system. One of the advantages of FCM is that we only need to develop various numerical algorithms for solving each DP in which the use of FIM completely avoids the well known roundoff-discretization error problem in using traditional finite difference method (FDM). Since the error of FIM depends on the choice of numerical quadrature formula, it can achieve much higher accuracy than FDM. Both error estimation and numerical examples given in the following confirm this distinct advantage. In addition, this FCM-FIM method is flexible to handle more general sideways problems with coefficients varying in x and t and equation with additional source term.

The paper is organized as follows. In Section 2, we present the general framework of FCM-FIM and explain how to use the FIM to solve ODE and obtain the error estimate by Tikhonov regularization method. In Section 3, the FIM is employed to solve PDE for 1-D and 2-D, respectively. Using the finite quotient to approximate the time derivative, we transform the sideways problem into a system of ODEs, whose solution is obtained iteratively by using the FIM. Several examples are constructed in Section 4 to verify the effectiveness and accuracy of the proposed FCM-FIM.

2. General framework of FCM-FIM

2.1. Forward collocation method (FCM)

For given $\varphi(t)$ chosen from a set of admissible boundary profiles $\Phi \subset C^1(0,T)$, assume that under some natural conditions the DP has a unique solution denoted by $u[\varphi] = u(x,t;\varphi)$. If the solution $u(x,t;\varphi)$ satisfies the additional condition $\partial u/\partial x|_{x=L} = \chi(t)$ for some functions $\varphi(t) \in \Phi$, then $\varphi = \varphi(t)$ is said to constitute a solution of the sideways problem.

From the least-square functional

$$F(\varphi) = \left\{ \int_0^T \left[\frac{\partial u[\varphi]}{\partial x}\bigg|_{x=L} - \chi(t) \right]^2 dt \right\}^{1/2} = \left\| \frac{\partial u[\varphi]}{\partial x}\bigg|_{x=L} - \chi(t) \right\|_{L^2(0,T)}, \tag{9}$$

we consider the following minimization problem:

$$\varphi_* = \arg \inf_{\varphi \in \Phi} F(\varphi). \tag{10}$$