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Boundary augmented Lagrangian method for contact problems in linear elasticity

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ABSTRACT

An augmented Lagrangian method, based on the fixed point method and boundary variational formulations, is designed and analysed for frictionless contact problems in linear elasticity. Using the equivalence between the contact boundary condition and a fixed point problem, we develop a new iterative algorithm that formulates the contact problem into a sequence of corresponding linear variational equations with the Steklov–Poincaré operator. Both theoretical results and numerical experiments show that the method presented is efficient.

1. Introduction

Contact problems have a large variety of applications in solid mechanics, including unilateral and bilateral contact or normal compliance conditions with friction [1–4]. Due to their nonlinear boundary conditions, these problems are difficult to solve. In the last 50 years, variational inequalities were a powerful tool in the mathematical study of contact problems, as the complexity of the boundary conditions and the diversity of the constitutive equations lead to variational formulations of inequality type. This and the fast development of computing power have led to an increased attention in the field of numerical algorithms for the problem. Presently there are two typical approaches for the numerical solution of the contact problem. One is to start with the discrete problem by the finite element method (FEM) [1–5] or the boundary element method (BEM) [6–10] and obtain a linear complementary problem, which generally results in an optimization problem in finite dimensional space. The second approach to solve the problem is to use the Lagrange multiplier. Then the contact problem is transformed into a sequence of linear elasticity problems [11–17]. The development of new, fast, and reliable methods for the numerical simulation of contact problems is still an area of frequent research [18–26].

Recently the fixed point method, based on the projection technique, has been successfully applied to complementary problems such as contact problems in linear elasticity [12,13,22,25–27]. The main idea of this method is to transform the complementary conditions into a fixed point problem using projection, which is very useful in developing various iterative methods for solving the original problem. During the last ten years, a number of fixed point methods have been studied extensively [12,13,16,17,23,26]. In these methods, the problem has been formulated only by equality with a projection operator, and no other inequality constraint is needed. In comparison to other methods, the fixed point method seems much easier to implement using the FEM.

On the other hand, BEM has turned out to be an accurate and effective method for many partial differential equations, including linear elasticity. The main benefit of BEM is the significant reduction of expense mesh generation because the formulation of the problem is reduced to the boundary of the domain. In the case of contact problems, the key unknowns are displacement and stress on the contact boundary, which are considered primary variables in BEM and can be obtained directly [30,31]. Therefore, the BEM is more appropriate for contact problems [6–9,32–36]. However, little attention has been paid to the contact problem using the fixed point method and BEM up to now.

The main goal of this paper is to develop a boundary augmented Lagrangian method (BALM) to deal with frictionless contact problems, focusing on fixed point methods and the BEM. For the contact problem, we first use the projection technique to deal with the nonlinear boundary conditions and obtain an equivalent formulation. Next, the link of the boundary weak formulation with the corresponding original problem is given. An advantage of the formulation is that, as compared with other methods, there is no inequality constraint and only a boundary integral equation is needed, which is useful from both theoretical and numerical

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points of view. Thanks to these transformations, we then propose a BALM for the contact problem, which needs only the iteration for boundary values and the computation of the boundary variational problem. We can use properties of projection and Steklov–Poincaré operator (also known as the Dirichlet-to-Neumann mapping) to analyse the convergence of the method. Usually the augmented Lagrangian method (ALM) needs to solve a nonlinear problem in every iteration step, but the semismooth Newton method can be applied for the solution [11,12,14]. Numerical results show that our method is accurate and efficient.

The paper is organized as follows. In Section 2 we start with the classical frictionless contact problem and introduce the fixed formulation of the contact boundary conditions. We recall the boundary integral operators for the Steklov–Poincaré operator and obtain the boundary variational formulation in Section 3. In Section 4 we propose a BALM for the contact problem and give convergence analysis of the method, which shows unconditional monotone convergence for all positive parameters. In Section 5, we carry out two numerical examples to investigate the performance of our method, and finally a brief conclusion and perspectives are drawn in Section 6.

2. Setting of the problem

For the sake of simplicity, we consider the frictionless contact problem in an open and bounded domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz boundary $\Gamma = \partial\Omega$. This boundary Γ consists of three disjointed parts Γ_D, Γ_N and $\Gamma_C \neq \emptyset$, where Dirichlet, Neumann and frictionless contact conditions are prescribed. Let $\mathbf{n}$ and $\mathbf{t}$ be the normal and tangential vector fields on Γ, respectively. For given boundary traction $\hat{\mathbf{t}} \in (L^2(\Gamma_N))^2$ and given obstacle $g \in H^{1/2}(\Gamma_C)$, find the displacement $\mathbf{u}$ such that

$$-\mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{0} \quad \text{in } \Omega, \tag{2.1}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D, \tag{2.2}$$

$$\boldsymbol{\sigma}(\mathbf{u}))\mathbf{n} = \hat{\mathbf{t}} \quad \text{on } \Gamma_N, \tag{2.3}$$

$$\sigma_t(\mathbf{u}) = 0 \quad \text{on } \Gamma_C, \tag{2.4}$$

$$u_n \le g, \quad \sigma_n(\mathbf{u}) \le 0, \quad \sigma_n(\mathbf{u})(u_n - g) = 0 \quad \text{on } \Gamma_C. \tag{2.5}$$

where $\boldsymbol{\sigma}$ denotes the stress tensor and σ_n and σ_t are the normal contact traction and the tangential contact traction, respectively. In this paper, we adopt the following decomposition for the displacement and the stress vector fields:

$$\mathbf{u} = u_n\mathbf{n} + \mathbf{u}_t \quad \text{and} \quad \boldsymbol{\sigma}(\mathbf{u}))\mathbf{n} = \sigma_n(\mathbf{u}))\mathbf{n} + \boldsymbol{\sigma}_t(\mathbf{u}).$$

It has been proven in the theory of variational inequalities that this problem has a unique solution [21,22].

On the boundary Γ_C, the zone of the classical Dirichlet and Neumann boundary conditions is unknown in advance. Therefore, the main challenge in such problem is how to identify the boundary condition on Γ_C. In this paper we transfer the complementary conditions (2.5) to a fixed point problem [22,23,28,29,37,38]. Let us introduce the projection notation for $a \in \mathbb{R}$:

$$[a]_+ := \begin{cases} a & \text{if } a > 0, \\ 0 & \text{otherwise}. \end{cases}$$

Consequently, we obtain the following result for the contact boundary condition. For the benefit of the reader, we give the proof according to [22,23].

Lemma 2.1. *For all $\rho > 0$, the contact conditions (2.5) on Γ_C are equivalent to*

$$\sigma_n(\mathbf{u}) + [\rho(u_n - g) - \sigma_n(\mathbf{u})]_+ = 0 \quad \text{on } \Gamma_C. \tag{2.6}$$

Proof. Let u_n and $\sigma_n(\mathbf{u})$ such that (2.5) holds. From the condition $\sigma_n(\mathbf{u}) \le 0$ we have either $\sigma_n(\mathbf{u}) < 0$ or $\sigma_n(\mathbf{u}) = 0$. Suppose first that $\sigma_n(\mathbf{u}) < 0$. Then the condition $\sigma_n(\mathbf{u})(u_n - g) = 0$ implies that $u_n = g$. In this case, it holds

$$[\rho(u_n - g) - \sigma_n(\mathbf{u})]_+ = [-\sigma_n(\mathbf{u})]_+ = -\sigma_n(\mathbf{u}).$$

Then, suppose that $\sigma_n(\mathbf{u}) = 0$. The condition $u_n \le g$ can also be expressed as $[\rho(u_n - g)]_+ = 0$, so

$$[\rho(u_n - g) - \sigma_n(\mathbf{u})]_+ = [\rho(u_n - g)]_+ = -\sigma_n(\mathbf{u}).$$

On the other hand, let u_n and $\sigma_n(\mathbf{u})$ such that (2.6) holds. Note first that it implies $\sigma_n(\mathbf{u}) \le 0$. If $\sigma_n(\mathbf{u}) = 0$, then (2.6) can be rewritten as

$$[\rho(u_n - g)]_+ = 0,$$

which is equivalent to condition $u_n \le g$. Hence, the following condition also holds:

$$\sigma_n(\mathbf{u})(u_n - g) = 0$$

We now consider the case $\sigma_n(\mathbf{u}) < 0$. From (2.6), $[\rho(u_n - g) - \sigma_n(\mathbf{u})]_+ > 0$, so that in this case

$$-\sigma_n(\mathbf{u}) = [\rho(u_n - g) - \sigma_n(\mathbf{u})]_+ = \rho(u_n - g) - \sigma_n(\mathbf{u}),$$

from which comes $u_n = g$, so that all conditions (2.5) hold.□

3. Boundary weak formulation of the contact problem

To develop a boundary variational formulation that is suitable for the contact problem, we start with the Hilbert space defined as

$$\boldsymbol{V} := \{\mathbf{v} \in (H^1(\Omega))^2, \ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\}.$$

From Green's formula and (2.1)–(2.5) we obtain the variational formulation as follows:

$$\int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v})\, dx = \int_{\Gamma_N \cup \Gamma_C} \boldsymbol{\sigma}(\mathbf{u}))\mathbf{n} \cdot \mathbf{v}\, ds, \quad \forall \mathbf{v} \in \boldsymbol{V}. \tag{3.1}$$

As in [7,32–34], we introduce the single layer potential V, the double layer potential K, the adjoint double layer potential K' and the hypersingular integral operator W by

$$(V\boldsymbol{\phi})(\boldsymbol{x}) = \int_\Gamma \mathbf{G}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\phi}(\boldsymbol{y})\, ds_y, \quad V : (H^{-1/2}(\Gamma))^2 \to (H^{1/2}(\Gamma))^2,$$

$$(K\boldsymbol{\phi})(\boldsymbol{x}) = \int_\Gamma \mathcal{T}_{n_y}\mathbf{G}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\phi}(\boldsymbol{y})\, ds_y, \quad K : (H^{1/2}(\Gamma))^2 \to (H^{1/2}(\Gamma))^2,$$

$$(K'\boldsymbol{\phi})(\boldsymbol{x}) = \int_\Gamma \mathcal{T}_{n_x}\mathbf{G}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\phi}(\boldsymbol{y})\, ds_y, \quad K' : (H^{-1/2}(\Gamma))^2 \to (H^{-1/2}(\Gamma))^2,$$

$$(W\boldsymbol{\phi})(\boldsymbol{x}) = -\mathcal{T}_{n_x}\int_\Gamma \mathcal{T}_{n_y}\mathbf{G}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\phi}(\boldsymbol{y})\, ds_y, \quad W : (H^{1/2}(\Gamma))^2 \to (H^{-1/2}(\Gamma))^2,$$

where $\mathbf{G}(\boldsymbol{x},\boldsymbol{y})$ is the fundamental solution of the two-dimensional Lamé equation

$$\mathbf{G}(\boldsymbol{x},\boldsymbol{y}) = \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)}\left\{\log\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\boldsymbol{I} + \frac{\lambda+\mu}{\lambda+3\mu}\frac{(\boldsymbol{x}-\boldsymbol{y}) \otimes \boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^2}\right\},$$

and $\mathcal{T}_n$ is the boundary traction operator defined by $\mathcal{T}_n(\mathbf{u}) := \boldsymbol{\sigma}(\mathbf{u})|_\Gamma \mathbf{n}$.

Now, we introduce the Dirichlet-to-Neumann mapping on Γ [31,33,34,39]:

$$S : (H^{1/2}(\Gamma))^2 \to (H^{-1/2}(\Gamma))^2$$
$$\mathbf{u}|_\Gamma \mapsto \boldsymbol{\sigma}(\mathbf{u})|_\Gamma.$$