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Boundary element method for vibration analysis of two-dimensional anisotropic elastic solids containing holes, cracks or interfaces

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ABSTRACT

By using the anisotropic elastostatic fundamental solutions and employing the dual reciprocity method, a special boundary element method (BEM) was developed in this paper to perform elastodynamic analysis of anisotropic elastic plates containing holes, cracks or interfaces. The system of ordinary differential equations obtained for the vibration transient analysis was solved using the Houlbolt's algorithm and modal superposition method. These equations were reduced to the standard eigenproblem for free vibration, and a purely algebraic system of equations for steady-state forced vibration. Since the fundamental solutions used in the present BEM satisfy the boundary conditions set on the holes, cracks, or interfaces, no meshes are needed along these boundaries. With this special feature, the numerical examples presented in this paper show that to get an accurate result much fewer elements were used in the present BEM comparing with those in the traditional BEM or finite element method.

1. Introduction

Stress concentration induced by holes, cracks, or interfaces is usually the main cause for the local failure of a structure. Although analytical closed form solutions have been found long time ago for an anisotropic plate containing an elliptical hole, crack or interface [1–6], they are applicable only for an infinite domain with static loads. In the design of engineering structures, numerical simulations such as the finite element method and boundary element method, play an important role to deal with the problems of complicated structural geometry and complex static or dynamic loading conditions. To increase the accuracy and efficiency of the numerical simulation, suitable utilization of the existing analytical solutions sometimes will make a great improvement, which motivates the present study.

For the traditional boundary element method (BEM) of dynamic analysis, the fundamental solutions used in boundary integral equation (BIE) are based on the solutions of the dynamic body force, i.e., a unit impulse force, which is in terms of time variable. However, practically it is difficult to find the solution analytically for particular problems, such as anisotropic plates with cracks, holes, or interfaces subjected to an impulse force. Even though such fundamental solutions could be obtained, the domain integral would have been present in the BIE if the body forces or non-zero initial conditions do exist [7]. In view of this, Nardini and Brebbia [8] developed the dual reciprocity boundary element method (DRBEM) to deal with the problems which involve the domain integral. By using a series of distributed shape functions to approximate the source term, the domain integral can be transformed into the equivalent boundary integral numerically. The applications of this method have been successful in a wide range of fields [9]. Thus, the elastostatics fundamental solutions can be used for dynamic analysis if the inertia term of dynamic problems is treated as a general body force, which is the source term remained in the domain integral.

A lot of approximate functions, such as general radial basis function (RBF), spline, multiquadric and Gaussian types RBFs, have been discussed to increase the accuracy of DRBEM [10]. For the isotropic elastic problems the most commonly used function is the first order conical radial basis function. However, in the case of anisotropic materials the particular solutions derived by the functions described above are not easy to be coped with in closed form. Kögl and Gaul [11,12] suggested an alternative approach to choose a particular solution, which is employed in this study and will be discussed in detail in the related section. Recently, the dual reciprocity method and the radial integration method have also been successfully used to treat the domain integrals of shear deformable orthotropic cracked plates under transverse distributed loads [13].

By the benefit of dual reciprocity method, the boundary element method for vibration analysis of anisotropic plates

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containing holes, cracks, or interfaces can be improved substantially by using the static fundamental solution which exactly satisfies the traction-free boundary conditions of holes or cracks, or the interface continuity condition. Same as the associated elastostatic problems [14–16], through the use of these special fundamental solutions, no meshes are needed along the boundary of holes, cracks, or interfaces, and hence the accuracy and efficiency of BEM can be improved.

Three kinds of vibration analysis are considered in this study, which are free vibration, steady-state forced vibration and transient analysis. Free vibration problem is accomplished by the generalized eigenvalue equation, and steady state analysis subjected to periodic harmonic loading is solved from a purely algebraic system of equations. The solution of the transient analysis involves the integration in time, and hence a step-by-step time integration technique is needed to solve the system of ordinary differential equations set for the vibration problems. Many numerical methods are available for the direct integration of the equations of motion, such as the Wilson θ-method, Newmark family of method and the Houbolt's method etc. [17–19]. For DRBEM, the feature of stability and practicability of the Houbolt's algorithms has been discussed [9,20]. A conclusion about how the numerical damping inherent in the Houbolt's algorithm affects the higher modes and lessens the problems resulting from the deleterious complex eigenfunctions is drawn. For this reason, the Houbolt's algorithm is a good choice especially for DRBEM. Also, the modal superposition method is suggested to transform the global space into the modal space to decouple the individual modes and carry out the time integration using only selected modes. This can eliminate the instabilities caused by the higher modes and improve the stability of DRBEM. With these solution techniques applied to the boundary element developed in this study, several numerical examples were carried out to prove its accuracy and efficiency, such as free vibration of isotropic plates with different shapes, and forced vibrations of isotropic/anisotropic plates with a crack/hole/interface subjected to a harmonic load or a Heaviside-type load.

2. Boundary integral equation for anisotropic elastodynamics

In a fixed rectangular coordinate system x_i, $i=1, 2, 3$, let u_i, σ_{ij}, ε_{ij}, and b_i be, respectively, the displacement, stress, strain, and body force. The strain–displacement relation for small deformation, the constitutive law for linear anisotropic elastic materials, and the equations of motion can be written as

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}), \quad \sigma_{ij}=C_{ijkl}\varepsilon_{kl}, \quad \sigma_{ij,j}+b_i=\rho\ddot{u}_i, \quad i,\ j,\ k,\ l=1,\ 2,\ 3, \tag{1}$$

where repeated indices imply summation, a comma stands for differentiation with respect to the coordinate variable, a dot indicates differentiation with respect to time, ρ is the mass density, and C_{ijks} are the elastic constants assumed to be fully symmetric, i.e., $C_{ijks}=C_{jiks}=C_{ijsk}=C_{ksij}$.

The reciprocal theorem of Betti and Raleigh in terms of stresses and strains can be expressed as

$$\int_\Omega \sigma_{ij}\varepsilon^*_{ij}d\Omega=\int_\Omega \sigma^*_{ij}\varepsilon_{ij}d\Omega, \tag{2}$$

where σ_{ij}, ε_{ij} and σ^*_{ij}, ε^*_{ij}, $i, j=1, 2, 3$, are the stresses and strains induced by two different loading systems on the same elastic body whose region is denoted by Ω. The boundary integral equations for anisotropic elastodynamics can be derived from (2) using the equations stated in (1), which are [21]

$$c_{ij}(\boldsymbol{\xi})u_j(\boldsymbol{\xi})+\int_\Gamma[t^*_{ij}(\mathbf{x},\boldsymbol{\xi})\,u_j(\mathbf{x})-u^*_{ij}(\mathbf{x},\boldsymbol{\xi})\,t_j(\mathbf{x})]d\Gamma(\mathbf{x}) = -\int_\Omega u^*_{ij}(\mathbf{x},\boldsymbol{\xi})[\rho\ddot{u}_j(\mathbf{x})-b_j(\mathbf{x})]d\Omega(\mathbf{x}), \tag{3}$$

where Γ is the boundary of region Ω, $t_j(\mathbf{x})$ is the surface traction along the boundary, u^*_{ij} and t^*_{ij} are the fundamental solutions of displacements and tractions of static problems, and $c_{ij}(\boldsymbol{\xi})$ is the free term coefficient dependent on the location of $\boldsymbol{\xi}$, which equals to $\delta_{ij}/2$ for a smooth boundary and $c_{ij}=\delta_{ij}$ for an internal point. The symbol δ_{ij} is Kronecker delta.

Since the static fundamental solution was used, the inertia term remains in a domain integral of the boundary integral equation, and hence, Eq. (3) is not a pure boundary representation of the displacements and tractions. Using the dual reciprocity method proposed by [8], this domain integral can be transformed to a boundary integral by approximating the source term with a series of known body forces $f_j^{(p)}(\mathbf{x})$ multiplied by the unknown coefficients α_p as

$$\rho\ddot{u}_j(\mathbf{x})-b_j(\mathbf{x})\cong-\sum_{p=1}^{N_p}\alpha_p f_j^{(p)}(\mathbf{x}). \tag{4}$$

Considering the body force $f_j^{(p)}(\mathbf{x})$ with $\ddot{u}_j(\mathbf{x})=0$ for static problems, the boundary integral Eq. (3) leads to

$$\int_\Omega u^*_{ij}(\mathbf{x},\boldsymbol{\xi})f_j^{(p)}(\mathbf{x})d\Omega(\mathbf{x})=c_{ij}(\boldsymbol{\xi})u_j^{(p)}(\boldsymbol{\xi})+\int_\Gamma[t^*_{ij}(\mathbf{x},\boldsymbol{\xi})u_j^{(p)}(\mathbf{x}) -u^*_{ij}(\mathbf{x},\boldsymbol{\xi})t_j^{(p)}(\mathbf{x})]d\Gamma(\mathbf{x}), \tag{5}$$

which successfully transforms a domain integral into a boundary integral. Functions $u_j^{(p)}(\mathbf{x})$ and $t_j^{(p)}(\mathbf{x})$ are the particular solutions of displacement and traction fields associated with the body force $f_j^{(p)}(\mathbf{x})$. Substituting (4) and (5) into (3), we obtain

$$c_{ij}(\boldsymbol{\xi})u_j(\boldsymbol{\xi})+\int_\Gamma[t^*_{ij}(\mathbf{x},\boldsymbol{\xi})u_j(\mathbf{x})-u^*_{ij}(\mathbf{x},\boldsymbol{\xi})t_j(\mathbf{x})]d\Gamma(\mathbf{x}) = \sum_{p=1}^{N_p}\alpha_p\left\{c_{ij}(\boldsymbol{\xi})u_j^{(p)}(\boldsymbol{\xi})+\int_\Gamma[t^*_{ij}(\mathbf{x},\boldsymbol{\xi})u_j^{(p)}(\mathbf{x})-u^*_{ij}(\mathbf{x},\boldsymbol{\xi})t_j^{(p)}(\mathbf{x})]d\Gamma(\mathbf{x})\right\}. \tag{6}$$

3. Fundamental solutions for holes, cracks, and interfaces

The fundamental solutions are usually recognized as Green's functions for an infinite region. If no holes, cracks, or interfaces are considered, the fundamental solutions can be obtained from the Green's functions for a homogeneous infinite space. By using the Stroh complex variable formalism for anisotropic elasticity, the fundamental solution for two-dimensional anisotropic elastostatics can be found in [22], which is

$$[u^*_{ij}(\mathbf{x},\boldsymbol{\xi})]=2\mathrm{Re}\{[\mathbf{AF}(z)]^T\}, \quad [t^*_{ij}(\mathbf{x},\boldsymbol{\xi})]=2\mathrm{Re}\{[\mathbf{BF}_{,s}(z)]^T\}, \tag{7}$$

where Re denotes the real part of a complex number, the superscript T stands for the transpose, the subscript $(,s)$ is a differentiation with respect to the tangential direction $\mathbf{s}$ of the field point $\mathbf{x}$, $\mathbf{A}$ and $\mathbf{B}$ are the material eigenvector matrices, and $\mathbf{F}(z)$ is the complex function matrix calculated by

$$\mathbf{F}(z)=\frac{1}{2\pi i}<\ln(z_\alpha-\hat{z}_\alpha)>\mathbf{A}^T, \tag{8}$$

where i is the imaginary unit defined by $i^2=-1$. The angular bracket $<\ >$ stands for the diagonal matrix in which each component is varied according to its subscript, e.g. $<z_\alpha> = \mathrm{diag}[z_1,z_2,z_3]$. $z_\alpha,\hat{z}_\alpha$ are related to the field point $\mathbf{x}=(x_1,x_2)$ and