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# The pre/post equilibrated conditioning methods to solve Cauchy problems

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ABSTRACT

In the present paper, the inverse Cauchy problems of Laplace equation and biharmonic equation are transformed, by using the method of fundamental solutions (MFS) and the Trefftz method (TM), to the systems of linear equations for determining the expansion coefficients. Then, we propose three different conditioners together with the conjugate gradient method (CGM) to solve the resultant ill-posed linear systems. They are the post-conditioning CGM and the pre-conditioning CGM based on the idea of equilibrated norm for the conditioned matrices, as well as a minimum-distance conditioner. These algorithms are convergent fast and accurate by solving the inverse Cauchy problems under random noise.


## 1. Introduction

The inverse Cauchy problem is to solve the boundary value problem of elliptic type partial differential equations given by an incomplete set of Cauchy data on a partial portion of the boundary, which is a well-known highly ill-posed problem, and many numerical methods have been developed to solve this sort problem for its possible engineering applications. In the past, almost in the checks to the ill-posedness of the inverse Cauchy problem, the illustrating examples have led to that the inverse Cauchy problem is actually severely ill-posed. Ben Belgacem [1] has provided an answer to the ill-posedness degree of the inverse Cauchy problem by using the theory of kernel operators. The foundation of his proof is the Steklov–Poincaré approach introduced in [2]. When the inverse Cauchy problem is defined in an arbitrary plane domain we can use the method of fundamental solutions [3–7], or the Trefftz method [8]. No matter which method is used we eventually need to solve an ill-posed linear equations system to determine the expansion coefficients.

Nevertheless the conjugate gradient method (CGM) is a composite of simple and elegant ideas, it is the most prominent iterative method for solving positive definite linear equations system. However, it is vulnerable to noisy disturbance on an ill-posed system. In this paper we propose *equilibrated and minimum-distance conditioning* conjugate gradient method to solve the following ill-posed linear system:

$$\mathbf{Bx} = \mathbf{b}_0, \tag{1}$$

where $\det(\mathbf{B}) \neq 0$ and $\mathbf{B} \in \mathbb{R}^{n\times n}$ may be an ill-conditioned, and generally unsymmetric coefficient matrix. The solution of such an ill-posed system of linear equations is an important issue for many physical and engineering problems. In practice, in the linear equations which arise in scientific problems, the data $\mathbf{b}_0$ are rarely given exactly; instead, noises in $\mathbf{b}_0$ are unavoidable due to the measurement error. Therefore, we may encounter the problem that the numerical solution of an ill-posed linear equations system may deviate from the exact one to a great extent, when $\mathbf{B}$ is ill-conditioned and $\mathbf{b}_0$ is polluted by noise.

It is well known that the direct methods and the iterative methods can be adopted to solve linear system (1). The former is widely employed when the order of the coefficient matrix $\mathbf{B}$ is not too large and is usually regarded as robust methods. The memory and the computational requirements for solving a large scale linear system may seriously challenge the most efficient direct solution methods available today. Currently, it is popular to use an iterative method to solve a large scale linear system. The reason is that iterative methods are easier to implement efficiently on high performance computers than direct methods. The application of iterative methods to determine meaningful approximate solution of Eq. (1) has previously been considered by applying the conjugate gradient method (CGM) to solve the normal equation associated with non-symmetric linear system (1); see Hanke [9] and Saad [10] and references therein.

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The approaches to the ill-posed linear problems can be categorized into three main classes: (a) regularizations of Eq. (1), (b) regularized algorithms to solve Eq. (1), and (c) a better pre-conditioning and/or post-conditioning to Eq. (1). In the splitting method, the matrix preconditioning technique is based on an approximation of the inverse of the coefficient matrix, where we assume that $\mathbf{B}=\mathbf{M}-\mathbf{N}$, and associate it with an iterative method:

$$\mathbf{x}_{k+1}=\mathbf{x}_k+\mathbf{M}^{-1}(\mathbf{b}_0-\mathbf{B}\mathbf{x}_k). \tag{2}$$

Here $\mathbf{M}^{-1}$ plays the role of a preconditioner. The more $\mathbf{M}$ resembles $\mathbf{B}$, the faster the iterative method will converge. One of the natural and simplest ways for the choice of the preconditioner is a diagonal of the coefficient matrix, such as the Jacobi method. However, it usually makes no remarkable reduction of the iteration number.

The scaling of linear algebraic equations is an important topic and has a long history for the development of scaling techniques. A matrix is equilibrated if all its rows or columns have the same norm, and under this condition the matrix is better conditioned. Really, there are theories of optimal scaling proposed by Bauer [11], van der Sluis [12], Watson [13], and Gautschi [14]. The problem is the search for some suitable diagonal matrices $\mathbf{D}_1$ and $\mathbf{D}_2$, such that the condition number of $\mathbf{D}_1\mathbf{B}\mathbf{D}_2$ is reduced as much as possible [15,16].

The CGM was used by Háo and Lesnic [17] and Jin [18] to solve the inverse Cauchy problem for Laplace's equation. This paper is a continuation of these efforts, which is organized as follows. The conjugate gradient method (CGM) and a preconditioned CGM for solving a positive definite linear equations system are introduced in Section 2. In Section 3 we describe three simple and direct equilibrated and minimum-distance conditioner methods for the solution of an ill-posed linear equations system. Section 4 devotes to use the post-conditioning CGM (PoCGM), pre-conditioning CGM (PrCGM) and the minimum-distance CGM (MDCGM) proposed in Section 3 together with the method of fundamental solutions (MFS) and the Trefftz method (TM) to solve inverse Cauchy problems of Laplace equation and biharmonic equation. Finally, the conclusions are drawn in Section 5. As we know the equilibrated and minimum-distance conditioners have not yet been exploited in the context of inverse Cauchy problems.

## 2. A preconditioned conjugate gradient method

A measure of the ill-posedness of Eq. (1) can be performed by using the condition number of $\mathbf{B}$ [19]:

$$\text{cond}(\mathbf{B})=\|\mathbf{B}\|_F\|\mathbf{B}^{-1}\|_F, \tag{3}$$

where $\|\mathbf{B}\|_F$ denotes the Frobenius norm of $\mathbf{B}$. Throughout this paper, the Euclidean norm is used for vector and the Frobenius norm is used for matrix, unless specified otherwise.

For every matrix norm $\|\bullet\|$ we have $\rho(\mathbf{B})\le\|\mathbf{B}\|$, where $\rho(\mathbf{B})$ is a radius of the spectrum of $\mathbf{B}$. The Householder theorem states that for every $\varepsilon>0$ and every matrix $\mathbf{B}$, there exists a matrix norm $\|\mathbf{B}\|$ depending on $\mathbf{B}$ and $\varepsilon$ such that $\|\mathbf{B}\|\le\rho(\mathbf{B})+\varepsilon$. Anyway, the spectral condition number $\rho(\mathbf{B})\rho(\mathbf{B}^{-1})$ can be used as an estimation of the condition number of $\mathbf{B}$ by

$$\text{cond}(\mathbf{B})=\frac{\max_{\sigma(\mathbf{B})}|\lambda|}{\min_{\sigma(\mathbf{B})}|\lambda|}, \tag{4}$$

where $\sigma(\mathbf{B})$ is the collection of all the eigenvalues of $\mathbf{B}$. Turning back to the Frobenius norm we have

$$\|\mathbf{B}\|_F\le\sqrt{n}\max_{\sigma(\mathbf{B})}|\lambda|. \tag{5}$$

In particular, for the symmetric case $\rho(\mathbf{B})\rho(\mathbf{B}^{-1})=\|\mathbf{B}\|_2\|\mathbf{B}^{-1}\|_2$.

Instead of Eq. (1), we can solve the normal equation

$$\mathbf{C}\mathbf{x}=\mathbf{b}, \tag{6}$$

where

$$\mathbf{C}:=\mathbf{B}^{\mathrm{T}}\mathbf{B}>\mathbf{0}, \tag{7}$$

$$\mathbf{b}:=\mathbf{B}^{\mathrm{T}}\mathbf{b}_0. \tag{8}$$

The conjugate gradient method (CGM), which is used to solve Eq. (6), is summarized as follows [20]:

(i) Give an initial $\mathbf{x}_0$.
(ii) Calculate $\mathbf{r}_0=\mathbf{b}-\mathbf{C}\mathbf{x}_0$ and $\mathbf{p}_1=\mathbf{r}_0$.
(iii) For $k=1,2,\ldots$, we repeat the following iterations:

$$\begin{aligned}
\alpha_k&=\frac{\|\mathbf{r}_{k-1}\|^2}{\mathbf{p}_k^{\mathrm{T}}\mathbf{C}\mathbf{p}_k},\\
\mathbf{x}_k&=\mathbf{x}_{k-1}+\alpha_k\mathbf{p}_k,\\
\mathbf{r}_k&=\mathbf{b}-\mathbf{C}\mathbf{x}_k,\\
\beta_k&=\frac{\|\mathbf{r}_k\|^2}{\|\mathbf{r}_{k-1}\|^2},\\
\mathbf{p}_{k+1}&=\mathbf{r}_k+\beta_k\mathbf{p}_k.
\end{aligned} \tag{9}$$

If $\mathbf{x}_k$ converges according to a given stopping criterion, such that,

$$\|\mathbf{r}_k\|<\varepsilon, \tag{10}$$

then stop; otherwise, go to step (iii). The norm used for $\mathbf{r}_k$ is the Euclidean norm.

We can evaluate the cost of computing the approximate solution $\mathbf{x}_k$ by using the CGM with $k$ steps. At each iterative step the CGM requires the matrix–vector multiplication of $n\times n$-matrix and $n$-vector two times and the inner products of two $n$-vectors two times. This portion is with totally $k(2n+2)$ multiplications. Dividing by the total number of steps $k$ we can obtain that each step requires $2n+2$ multiplications on the average. It is known that the CGM converges within $n$ steps. So we can say that the computational complexity of CGM is $\mathcal{O}(n^2)$.

It is well known that the convergence speed of CGM depends on the distribution of the eigenvalues of the coefficient matrix $\mathbf{C}$ as shown by the following formula [10,21]:

$$\|\mathbf{x}_k-\mathbf{x}^*\|\le 2\left(\frac{\sqrt{\kappa_2}-1}{\sqrt{\kappa_2}+1}\right)^k\|\mathbf{x}_0-\mathbf{x}^*\|, \tag{11}$$

where $\mathbf{x}^*$ is the exact solution of Eq. (6), and $\kappa_2=\|\mathbf{C}\|_2\|\mathbf{C}^{-1}\|_2$.

When the coefficient matrix is typically extremely ill-conditioned, the convergence of CGM can be unacceptably slow. In this case, the CGM is not competitive without a good preconditioner. That is, the preconditioning technique is a key ingredient for the success of CGM in applications. The idea of preconditioning technique is based on the consideration of the linear system with the same solution as the original equation. The problem is that each preconditioning technique is suited for a different type of problem. Until now no robust preconditioning technique appears for all or at least much types of problems. Finding a good preconditioner to solve a given large scale linear system is often viewed as a combination of art and science.

To improve the convergence speed of iterative methods, an appropriate preconditioner can be incorporated. Based on the survey by Benzi [22], a good preconditioner should meet the following requirements: (1) the preconditioned system should be easy to solve, and (2) the preconditioner should be cheap to construct and apply. In order to increase the convergence speed of CGM, we require to reduce the condition number of $\mathbf{C}$ as shown in Eq. (11). For the purpose of comparison the preconditioned CGM